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The probability of selecting k edge-disjoint Hamilton cycles in the complete graph

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Abstract

Let $H_1, \ldots, H_k$ be Hamilton cycles in K_n, chosen independently and uniformly at random. We show, for $k = o(n^{1/100})$, that the probability of $H_1, \ldots, H_k$ being edge-disjoint is $(1 + o(1))e^{-2\binom{k}{2}}$. This extends a corresponding estimate obtained by Robbins in the case $k = 2$.

1 Introduction

A classical problem in elementary combinatorics is to show that the number of derangements of an n element set (recall that a derangement is a permutation with no fixed points) is $(1 + o(1))\frac{n!}{e}$. This problem can equivalently be formulated in graph theoretic language as follows: let $K_{n,n}$ be the complete bipartite graph with each part of size n. Let M_1 and M_2 be two perfect matchings of $K_{n,n}$, chosen independently and uniformly at random. Then, the probability that $M_1 \cap M_2 = \emptyset$ is $(1 + o(1))\frac{1}{e}$.

This formulation immediately suggests the following question: suppose that $M_1, \ldots, M_k$ are k perfect matchings of $K_{n,n}$, each of which is chosen independently and uniformly at random. What is the probability that all of the M_i's are edge-disjoint? Using (nowadays) standard estimates on the permanent of the (bipartite) adjacency matrix of a d-regular bipartite graph, one can readily show that (for k which is not too large compared to n), the answer to this question is $(1 + o(1))e^{-\binom{k}{2}}$ – we leave this as an exercise for the interested reader. Of course, one may ask the same question with perfect matchings replaced by any other graph, and $K_{n,n}$ replaced by some other 'ground graph'

Problem 1.1. *Let H be a graph on at most n vertices, and let G be a graph that contains at least one copy of H. Let $X_1, \ldots, X_k$, $k \geq 2$, be i.i.d random variables, each of which outputs a copy of H in G, distributed uniformly at random. What is the probability $p(G, H, k)$ that all the copies X_i are edge-disjoint?*

For $G = K_n$ (the complete graph on n vertices), $H = C_n$ (a simple cycle on n vertices, also known as a *Hamilton cycle*), and $k = 2$, it was shown in [3] using a clever inclusion-exclusion argument that $p(G, H, 2) = (1 + o(1))e^{-2}$. As in the case of perfect matchings, it is natural to ask for $p(G, H, k)$ for $k > 2$. Unfortunately, it seems rather hard to extend the argument of [3] to larger values of k (in fact, even a possible extension to $k = 3$ seems quite involved). In this note, using a completely different argument, we resolve this problem for all values of k up to some small polynomial in n. Specifically, we prove the following theorem.

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Theorem 1.2. *Let $k = o(n^{1/100})$, and let $H_1, \ldots, H_k$ be Hamilton cycles in K_n, each of which is chosen independently and uniformly at random. Then, the probability that all the H_i's are edge disjoint is*

$$(1 + o(1))e^{-2\binom{k}{2}}.$$

Remark 1.3. *In order to keep the exposition simple, and since our approach anyway does not seem to work for values of k larger than $\sqrt{n}$, we did not make any effort to optimize the upper bound on k in the above theorem.*

1.1 Outline of the proof

By Bayes' rule, it suffices (see Section 3 for details) to show that the number of Hamilton cycles in any graph obtained by removing i edge-disjoint Hamilton cycles from K_n is $(1+o(1/k)) \cdot e^{-2i} \cdot (n-1)!/2$ – this is the content of our main technical lemma (Lemma 2.7). Our proof of this lemma consists of providing an algorithm to generate all Hamilton cycles in a given graph G (see Section 2.2), and then using standard estimates on the number of perfect matchings in bipartite graphs (Corollary 2.5, Lemma 2.6), as well as standard concentration inequalities (Theorem 2.1), in order to analyze the number of distinct Hamilton cycles our algorithm can output.

Roughly speaking, our algorithm generates Hamilton cycles as follows: for a sufficiently large integer ℓ, divide the vertices of G into ℓ parts of size n/ℓ (for the sake of this discussion, we assume that n is divisible by ℓ); choose a perfect matching between parts i and $i+1$ for $1 \leq i \leq \ell - 1$ to obtain a collection of n/ℓ (oriented) paths of length $\ell - 1$, and finally, extend (if possible), this collection of oriented paths to an oriented Hamilton cycle of G. In Claim 2.9, we show using a standard concentration of measure argument that most ways to partition the vertices satisfy a certain 'goodness' property – the contribution to our enumeration coming from partitions not satisfying this property is so small that it may be ignored. On the other hand, for partitions satisfying this goodness property, we are able to effectively leverage standard estimates on the number of perfect matchings in bipartite graphs to provide an asymptotically correct estimate of the number of choices available to our algorithm in subsequent steps.

2 Tools and auxiliary results

In this section, we collect some tools and auxiliary results to be used in the proof of our main result.

2.1 McDiarmid's inequality

We will make use of the following concentration inequality due to McDiarmid (see [2], Section 3.2).

Theorem 2.1. *Let S_n denote the symmetric group on n elements and let $f : S_n \to \mathbb{R}$ be such that $|f(\pi) - f(\pi')| \leq u$ whenever π' can be obtained from π by a single transposition. If π is chosen uniformly at random from S_n, then*

$$\Pr\left[|f(\pi) - \mathbb{E}(f)| \geq t\right] \leq 2\exp\left(-\frac{2t^2}{nu^2}\right).$$

2.2 A procedure to find all Hamilton cycles in a graph G

In this section, we describe a procedure to find all Hamilton cycles in a given graph G. Later, in Section 2.4, we will estimate (from below and above) the number of distinct Hamilton cycles that this procedure can output.

1. Fix any positive integer ℓ (possibly depending on n). Let $r = n \mod \ell$, so that $0 \le r \le \ell-1$. Let $[n] = V_1 \cup \ldots \cup V_\ell$ be any partition of $[n]$, where the first r parts have size $t_c := \lceil \frac{n}{\ell} \rceil$, and the last $\ell - r$ parts have size $t_f := \lfloor \frac{n}{\ell} \rfloor$.
2. If $r \neq 0$, designate a 'root' $v^* \in V_r$. If $r = 0$, designate a 'root' v^* in V_1.
3. For each $j \in [1, r-1] \cup [r+1, \ell-1]$, let $B_j := G[V_j, V_{j+1}]$. If $r \neq 0$, let $B_r := G[V_r \setminus \{v^*\}, V_{r+1}]$ and $B_\ell := G[V_\ell \cup \{v^*\}, V_1]$; if $r = 0$, let $B_\ell := G[V_\ell, V_1]$.
4. For each $1 \le j \le \ell - 1$, choose a perfect matching M_j of B_j, and observe that $\cup_j M_j$ is a collection of t_c vertex disjoint paths, of which t_f have length ℓ and $t_c - t_f$ have length r (by the length of a path, we mean the number of vertices in it). Let $\mathcal{P} := \{P_1, \ldots, P_{t_c}\}$ denote the obtained collection of paths, and orient each path such that the vertex in V_1 is the first vertex.
5. Finally, using only the edges in B_ℓ (directed from V_ℓ to V_1), find (if possible) a rooted, oriented Hamilton cycle in G, which is rooted at v^* and contains all the paths in $\mathcal{P}$ as oriented segments.

Let $\mathcal{H}_{r,o}(\ell)$ denote the collection of rooted, oriented Hamilton cycles in G obtained by running the above procedure (with some fixed positive integer ℓ) for all possible choices of partitions in Step 1, all possible choices of the root in Step 2, all possible choices of the perfect matchings in Step 4, and all possible choices of the compatible rooted, oriented Hamilton cycle in Step 5.

Lemma 2.2. *For every positive integer ℓ, the collection $\mathcal{H}_{r,o}(\ell)$ contains every rooted, oriented Hamilton cycle of G exactly once.*

Proof. Fix a rooted, oriented Hamilton cycle H in G. There is exactly one partition of the vertices in Step 1 compatible with H – indeed, the root v^* must belong to $V_{\max\{r,1\}}$, and following the Hamilton cycle from the root along its orientation determines the partition of the vertices. Once this is done, note that the choice of perfect matchings (equivalently, the collection of oriented paths $\mathcal{P}$) in Step 4 is automatically determined by the edges present in the Hamilton cycle. Finally, given this collection of paths, there is exactly one choice of edges in Step 5 which is compatible with H. ☐

Let $\mathcal{H}(\ell)$ be the collection of Hamilton cycles in G, obtained from $\mathcal{H}_{r,o}(\ell)$ by forgetting the root and the orientation. Since for any Hamilton cycle in G, there are exactly n ways to choose a root for it, and exactly 2 ways to orient it, we have:

Observation 2.3. *For every positive integer ℓ, the collection $\mathcal{H}(\ell)$ contains every Hamilton cycle of G exactly $2n$ times.*

2.3 The number of perfect matchings in bipartite graphs

In order to estimate the number of Hamilton cycles our procedure can output, we will need to estimate the number of perfect matchings in 'typical' bipartite graphs obtained by our procedure.

For bounding the number of perfect matchings in a bipartite graph from above, we use the following theorem due to Brégman (see e.g. [1], page 24) that relates the number of perfect matchings to the vertex-degrees in the graph.

Theorem 2.4. *(Brégman's Theorem) Let $G = (A \cup B, E)$ be a bipartite graph with both parts of the same size. Then, the number of perfect matchings in G is at most*

$$\prod_{a \in A} (d_G(a)!)^{1/d_G(a)}.$$

The following is an immediate corollary of Theorem 2.4 and Stirling's approximation.

Corollary 2.5. *Let H be a spanning subgraph of $K_{m,m}$ with maximum degree at most $D \le m/2$. Then, the number of perfect matchings in $K_{m,m} \setminus H$ is at most*

$$e^{O(D \log m/m)} \cdot m! \cdot e^{-|E(H)|/m}.$$

Proof. By applying Theorem 2.4 to $G := K_{m,m} \setminus H$ and using the fact that $s! = (1+O(1/s))\sqrt{2\pi s}\left(\frac{s}{e}\right)^s$, one obtains that the number of perfect matchings in $K_{m,m} \setminus H$ is at most

$$\left(\prod_{a\in A}\left(1+O\left(\frac{1}{m-D}\right)\right)^{1/(m-D)}\right)\cdot\left(\prod_{a\in A}(\sqrt{2\pi m})^{1/(m-D)}\right)\cdot\left(\prod_{a\in A}\left(\frac{m-d_H(a)}{e}\right)\right).$$

Using the assumption $D \le m/2$, the first term can be estimated by:

$$\begin{aligned}\prod_{a\in A}\left(1+O\left(\frac{1}{m-D}\right)\right)^{1/(m-D)} &\le e^{O(m/(m-D)^2)}\\ &\le e^{O(1/m)}.\end{aligned}$$

The second term can be estimated by:

$$\begin{aligned}\prod_{a\in A}(\sqrt{2\pi m})^{1/(m-D)} &= \sqrt{2\pi m}\cdot(\sqrt{2\pi m})^{D/(m-d)}\\ &\le \sqrt{2\pi m}\cdot e^{O(D\log m/(m-D))}\\ &\le \sqrt{2\pi m}\cdot e^{O(D\log m/m)}.\end{aligned}$$

The third term can be estimated by:

$$\begin{aligned}\prod_{a\in A}\left(\frac{m}{e}\cdot\left(1-\frac{d_H(a)}{m}\right)\right) &\le \left(\frac{m}{e}\right)^m\cdot\prod_{a\in A}\exp\left(-\frac{d_H(a)}{m}\right)\\ &\le \left(\frac{m}{e}\right)^m\cdot\exp\left(-\frac{|E(H)|}{m}\right).\end{aligned}$$

Combining everything, and using Stirling's approximation once again, we get the upper bound:

$$e^{O(D\log m/m)}\cdot\left(\sqrt{2\pi m}\left(\frac{m}{e}\right)^m\right)\cdot e^{-|E(H)|/m} \le e^{O(D\log m/m)}\cdot m!\cdot e^{-|E(H)|/m},$$

as desired. ☐

The next lemma provides a nearly matching lower bound on the number of perfect matchings in 'almost complete' balanced bipartite graphs.

Lemma 2.6. *Let H be a spanning subgraph of $K_{m,m}$ with $|E(H)| < \frac{m}{4}$. Then, the number of perfect matchings in $K_{m,m} \setminus H$ is at least*

$$\left(1-O\left(\frac{|E(H)|^2}{m^2}\right)\right)\cdot m!\cdot e^{-|E(H)|/m}.$$

Proof. Represent $G := K_{m,m} \setminus H$ as $G = (A \cup B, E)$, and label the vertices of $A = \{v_1, \dots, v_m\}$ in such a way that all the $q \le m/4$ vertices in A, which are not isolated in H, are labeled as $v_1, v_2, \dots, v_q$. For each $1 \le i \le q$, let $d_i := d_H(v_i)$.

We will construct perfect matchings of G by manually pairing each vertex in A with a vertex in B. For this, note that there are at least $m - d_1$ ways to choose a vertex in B to pair with v_1. Having chosen such a vertex, there are at least $m - d_2 - 1$ ways to choose a vertex in B, different from the one chosen in the previous step, to pair with v_2. In general, for $1 \le i \le q$, there are at least $m-d_i-(i-1)$ ways to choose a vertex in B, different from the ones chosen in the first $i-1$ steps, to pair with v_i. Having matched the first q vertices, note that all of the remaining vertices in A have edges (in G) to all of the vertices in B and hence, the number of ways in which we can find a vertex in B to pair with v_i for $i>q$ is exactly $m-(i-1)$. Since each sequence of choices gives a different perfect matching, it follows that the number of perfect matchings in G obtained in this manner is at least

$$\begin{aligned}\left(\prod_{i=1}^{q}(m-d_i-(i-1))\right)\prod_{i=q+1}^{m}(m-(i-1)) &= m!\cdot\prod_{i=1}^{q}\frac{m-d_i-i+1}{m-i+1}\\ &= m!\cdot\prod_{i=1}^{q}\left(1-\frac{d_i}{m-i+1}\right)\\ &\geq m!\left(1-\sum_{i=1}^{q}\frac{d_i}{m-i+1}\right)\\ &\geq m!\left(1-\sum_{i=1}^{q}\frac{d_i}{m-q}\right)\\ &= m!\left(1-\frac{|E(H)|}{m-q}\right),\end{aligned}$$

where the third line uses the elementary inequality $\prod_{i=1}^{n}(1-x_i)\geq 1-\sum_{i=1}^{n}x_i$, valid for $x_1,\dots,x_n\geq 0$. Next, using the numerical inequality $(1-x)^{-1}\leq 1+2x$, valid for $x\in[0,1/2]$, we have

$$\begin{aligned}m!\left(1-\frac{|E(H)|}{m-q}\right) &\geq m!\left(1-\frac{|E(H)|}{m}\cdot\left(1+\frac{2q}{m}\right)\right)\\ &= m!\left(1-\frac{|E(H)|}{m}\right)\left(1-\frac{2q|E(H)|}{m(m-|E(H)|)}\right)\\ &\geq m!\left(1-\frac{|E(H)|}{m}\right)\left(1-3\frac{|E(H)|^2}{m^2}\right),\end{aligned}$$

where the last equality uses that $q\leq|E(H)|<m/4$. Finally, using the numerical inequality $1-x\geq e^{-x}(1-x^2)$, valid for $x\in[0,1]$, we can bound the right hand side from below by

$$\begin{aligned}m!\cdot e^{-|E(H)|/m}\left(1-\frac{|E(H)|^2}{m^2}\right)\left(1-3\frac{|E(H)|^2}{m^2}\right) &\geq m!\cdot e^{-|E(H)|/m}\left(1-10\frac{|E(H)|^2}{m^2}\right)\\ &= \left(1-O\left(\frac{|E(H)|^2}{m^2}\right)\right)m!\cdot e^{-|E(H)|/m}.\end{aligned}$$ □

2.4 The number of Hamilton cycles obtained by our procedure

In this section, we present the key step in the proof of our main theorem – a near-optimal estimate on the number of Hamilton cycles in a graph G obtained by deleting $i<k$ edge-disjoint Hamilton cycles from K_n. Specifically, we prove the following lemma:

Lemma 2.7. *Let $k=o(n^{1/100})$, and let $H_1,\dots,H_k$ be i.i.d. random variables, each of which outputs a Hamilton cycle of K_n, chosen uniformly at random. For each $1\leq i\leq k$, let $\mathcal{E}_i$ be the event "$E(H_i)\cap(\cup_{j<i}E(H_j))=\emptyset$" i.e. no edge of H_i is present in H_j, for any $j<i$. Then, for every $0\leq i\leq k-1$*

$$\Pr\left[\mathcal{E}_{i+1}\mid\mathcal{E}_i\cdots\mathcal{E}_1\right]=\exp\left(\pm O\left(\frac{1}{k^2\log^4 n}\right)\right)\cdot\exp(-2i).$$

To prove this lemma, we will analyze the procedure for generating all Hamilton cycles of G given in Section 2.2. We will need the following two preliminary claims.

The first claim concerns the number of partitions in Step 1 of Section 2.2.

Claim 2.8. *The number of partitions $\mathcal{V}$ of $[n]$ into r sets $V_1, \dots, V_r$ of size t_c and $\ell - r$ sets $V_{r+1}, \dots V_\ell$ of size t_f, together with a designated vertex $v^* \in V_{\max\{r,1\}}$ is $\frac{n! \cdot t_c}{(t_c!)^r (t_f!)^{\ell-r}}$.*

Proof. Indeed, there are $\frac{n!}{(t_c!)^r (t_f!)^{\ell-r}}$ ways of choosing an (ordered) partition with the given sizes, and t_c ways of choosing a designated vertex from $V_{\max\{r,1\}}$. $\square$

Let G be a graph obtained from K_n by removing i edge-disjoint Hamilton cycles, and fix $\ell \leq \sqrt{n}$. For a partition of $[n]$ into ℓ parts as above, let $\{B_j\}_{1 \leq j \leq \ell}$ denote the collection of bipartite graphs constructed in Step 3 of Section 2.2. We claim that, for most partitions, the number of edges missing from each B_j is close to its expectation (for a uniformly random partition).

Claim 2.9. *Let f_j be the number of missing edges in B_j. Then, for all sufficiently large n, the number of partitions $\mathcal{V}$ of $[n]$ for which $|f_j - \frac{2 \cdot i \cdot n}{\ell^2}| \leq 2n^{2/3}$ for every $1 \leq j \leq \ell$ is at least*

$$(1 - e^{-n^{3/100}}) \frac{n!}{(t_c!)^r (t_f!)^{\ell-r}}.$$

Proof. Let $\sigma \in S_n$ be a uniformly random permutation of $[n]$. Let V_1 be the image of $[1, |V_1|]$ under σ, V_2 be the image of the next $|V_2|$ elements in $[n]$ under σ, and so on. For $1 \leq j \leq \ell$, let f_j be the number of missing edges in B_j, and observe that

$$\mu_j := \mathbb{E} f_j = \frac{2 \cdot i \cdot n}{\ell^2} \pm O\left(\frac{i}{\ell}\right) = \frac{2 \cdot i \cdot n}{\ell^2} \pm o(n^{2/3}),$$

where the final inequality uses the assumption $i \leq k = o(n^{1/100})$. Next, since $\cup_{j \leq i} E(H_i)$ is a $2i$-regular graph, it follows that a single transposition of σ can change f_j by at most $2i$. Therefore, by Theorem 2.1,

$$\Pr[|f_j - \mu_j| \leq n^{2/3}] \leq 2e^{-2n^{4/3}/(4 \cdot n \cdot i^2)} = o(e^{-n^{3/100}}/\ell),$$

where the final inequality uses the assumption $i \leq k = o(n^{1/100})$. Applying the union bound for $1 \leq j \leq \ell$ shows that the number of permutations giving rise to 'good partitions' (i.e. those satisfying the assumptions of the lemma) is at least $\left(1 - e^{-n^{3/100}}\right) n!$. Finally, since each partition corresponds to $(t_c!)^r (t_f!)^{\ell-r}$ distinct permutations, we get the desired conclusion. $\square$

With these two claims in hand, we can prove Lemma 2.7.

Proof of Lemma 2.7. Let $H_1, \dots, H_i$ be any edge-disjoint Hamilton cycles in K_n, and let G be the graph obtained from K_n by removing $\cup_{j \leq i} E(H_j)$. We wish to count the number of Hamilton cycles in G, and we will do so by analyzing the procedure in Section 2.2. For the rest of this proof, we fix $\ell = \lceil (k \log n)^4 \rceil = o(n^{1/20})$.

First, note that the number of Hamilton cycles in G obtained by our procedure, starting from a partition in Step 1 which does not satisfy the conclusion of Claim 2.9 is negligible for our purposes. Indeed, by Claim 2.9, the number of such partitions is at most $e^{-n^{3/100}} \cdot n!/((t_c!)^r \cdot (t_f!)^{\ell-r})$, and once we fix such a partition, the number of choices available in Steps 2-5 is at most $t \cdot (t_c!)^r \cdot (t_f!)^{\ell-r}$. Hence, the number of Hamilton cycles that can be obtained in this manner is at most $t \cdot n! \cdot e^{-n^{3/100}} = o(n! \cdot e^{-2i}/\mathrm{poly}(n))$, for $i \leq k = o(n^{1/100})$. Therefore, it suffices to analyze the contribution of partitions satisfying the conclusion of Claim 2.9.

For any such partition $V = V_1 \cup \cdots \cup V_\ell$, there are exactly t_c choices in Step 2.

Moreover, for each such realisation of Steps 1-3, by Corollary 2.5 and Claim 2.9, the number of perfect matchings of B_j is at most either (depending on the value of j)

$$t_c! \cdot \exp\left(-\frac{2 \cdot i}{\ell}\right) \cdot \exp(O(\ell \cdot n^{-1/3})),$$

or the same expression with t_c replaced by t_f. Similarly, by Lemma 2.6 and Claim 2.9, the number of perfect matchings of B_j is at least either (depending on the value of j)

$$t_c! \cdot \exp\left(-\frac{2 \cdot i}{\ell}\right) \cdot \exp\left(-O\left(\frac{k^2}{\ell^2}\right)\right),$$

or the same expression with t_c replaced by t_f.

Since there are $r-1$ values of j for which the above bounds hold with t_c, and $\ell - r$ values of j for which the above bounds hold with t_f, and since $k^2/\ell^2 \gg \ell \cdot n^{-1/3}$, it follows that the number of collection of paths that can be obtained at the end of Step 4 is

$$(t_c!)^{r-1} \cdot (t_f!)^{\ell-r} \cdot \exp(-2i) \cdot \exp\left(\pm O\left(\frac{k^2}{\ell}\right)\right).$$

Finally, let us estimate the number of ways to extend any such collection of paths into a Hamilton cycle in Step 5. For this, we arbitrarily label the collection of paths obtained at the end of Step 4 by $1, \ldots, t_c$. This induces a natural labeling (given by which path the vertex participates in) of each part of the bipartite graph B_ℓ by the labels $1, \ldots, t_c$. The obtained labelled bipartite graph may be viewed as a directed graph (possibly with self loops) on t_c vertices as follows: identify vertices with the same label, and orient edges from the first part of B_ℓ to its second part. Observe that the the number of extensions available in Step 5 correspond precisely to the number of oriented Hamilton cycles of this directed graph.

Since the complete directed graph on t_c vertices has at most $(t_c-1)!$ oriented Hamilton cycles, it follows that there are at most $(t_c-1)!$ such extensions. For a nearly matching lower bound, we begin by noting that the number of oriented Hamilton cycles in the complete directed graph on t_c vertices containing a specific edge is at most $(t_c-2)!$. Since, by Claim 2.9 the directed graph corresponding to B_ℓ has at most $O(k \cdot n/\ell^2)$ missing edges, it follows that the number of oriented Hamilton cycles in this directed graph is at least

$$(t_c-1)! - O\left(\frac{k \cdot n}{\ell^2}\right)(t_c-2)! = (t_c-1)!\left(1 - O\left(\frac{k}{\ell}\right)\right).$$

To summarize, we have shown that the number of choices available in Steps 2-5, for any fixed choice of partition in Step 1 which satisfies the conclusion of Claim 2.9 is

$$t_c \cdot (t_c!)^{r-1} \cdot (t_f!)^{\ell-r} \cdot \exp(-2i) \cdot (t_c-1)! \exp\left(\pm O\left(\frac{k^2}{\ell}\right)\right) = (t_c!)^r \cdot (t_f!)^{\ell-r} \cdot \exp(-2i) \cdot \exp\left(\pm O\left(\frac{k^2}{\ell}\right)\right).$$

Combining this with the number of choices in Step 1, as given by Claim 2.9, we see that the contribution to the number of oriented, rooted Hamilton cycles from such partitions is

$$n! \cdot \exp(-2i) \cdot \exp\left(\pm O\left(\frac{k^2}{\ell}\right)\right) = 2n \cdot \frac{(n-1)!}{2} \cdot \exp(-2i) \cdot \exp\left(\pm O\left(\frac{1}{k^2 \log^4 n}\right)\right)$$

To conclude, recall that every (undirected, unrooted) Hamilton cycle is counted exactly $2n$ times by our procedure, and that K_n has $(n-1)!/2$ Hamilton cycles. □

3 Proof of the main theorem

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. The proof is a straightforward application of Lemma 2.7. Indeed, we wish to find the probability that all k of the chosen Hamiltonian cycles are edge disjoint, which is

$$\begin{aligned}
\Pr[\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_k] &= \Pr[\mathcal{E}_1]\Pr[\mathcal{E}_2 \mid \mathcal{E}_1]\ldots\Pr[\mathcal{E}_k \mid \mathcal{E}_{k-1}\ldots\mathcal{E}_2\mathcal{E}_1] \\
&= \prod_{i=0}^{k-1}\Pr[\mathcal{E}_{i+1} \mid \mathcal{E}_i\cdots\mathcal{E}_1] \\
&= \prod_{i=0}^{k-1}\exp\left(\pm O\left(\frac{1}{k^2\log^4 n}\right)\right)\cdot\exp(-2i) \\
&= \exp\left(\pm O\left(\frac{1}{k\log^4 n}\right)\right)\exp\left(-2\sum_{i=0}^{k-1} i\right) \\
&= (1+o(1))e^{-2\binom{k}{2}},
\end{aligned}$$

where the third line uses Lemma 2.7. □

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