Linear Algebra in Combinatorics problem set

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Here are a couple of Combinatorics problem, most of which can be solved using techniques from linear algebra. Initially I included hints for some problems in the statements, but I have decided to move all hints to a separate section at the end so that you can attempt the problems without looking at the hint first, if you want. Do use the hints if you get stuck though, many of these problems are very difficult!

If there are some words that are not explained in the problem that you do not know the meanings of, they should have been defined in the lecture notes. If this is not the case, don't hesitate to ask!

1 Town Problems

Problem 1.1. For a matrix A, we define $rk(A)$ to be the column rank, i.e the dimension of the subspace spanned by the columns of A. Prove the following facts:

(a) $rk(A) = rk(A^T)$. In other words, the row rank (dimension of the span of the rows) is the same as the column rank. (Note that A doesn't have to be a square matrix).

(b) Show that if A is a $k \times m$ matrix and B is an $m \times n$ matrix, we have that

$$rk(AB) \leq \min\{rk(A),\ rk(B)\}$$

Problem 1.2. (Eventown) In a town with n people, there is some number of clubs. Each club has an even number of members, and any two clubs have an odd number of members in common.

(a) Show that there can be at most n different clubs.

(b) Show that the bound n is achievable when n is odd.

(c) Show that when n is even, there can be at most $n - 1$ clubs. Furthermore, show that this is achievable.

Problem 1.3. In a town with n people, there are m red clubs $R_1, ..., R_m$ and m blue clubs $B_1, ..., B_m$. Assume that the clubs satisfy the following rules:

(a) $|R_i \cap B_i|$ is odd for every i

(b) $|R_i \cap B_j|$ is even for every $i \neq j$

Show that $m \leq n$.

Problem 1.4. In a town with n people, there is some number of clubs. Any two clubs have the same number of members in common. Show that there can be at most n different clubs.

Problem 1.5. Prove the result about Oddtown by considering vectors over $\mathbb{Q}$ instead of $\mathbb{F}_2$.

Problem 1.6. (difficult) In a town with n people, there are m clubs. Each club has an even number of members, and any two clubs have an even number of members in common. Show that $m \leq 2^{n/2}$.

2 Projective Planes

A *projective plane* consists of a set of points P and a set of lines L. Each lines contains some of the points in such a way that the following holds:

(a) For any two points, there is exactly one line which contains both points.

(b) For any two lines, there is exactly one point which is contained in both of them.

(c) There exists four points p_1, p_2, p_3, p_4 such that no line contains three of them.

The third condition is just a technical one which rules out some silly cases. Since the first two conditions are completely symmetric in points and lines, anything we can prove about projective planes will hold if we swap the roles of points and lines. This is known as point-line *duality.*

Given the duality between points and lines, it shouldn't come as a huge surprise that any finite projective plane must have the same number of lines and points. But, while it might seem plausible, this is a fact which needs a proof.

Problem 2.1. Assume we have a finite set of n points $P = \{p_1, ..., p_n\}$ and a set of m lines $L = \{l_1, ..., l_m\}$ each of which contains some of the points. Furthermore assume that no line contains all points.

(a) Show that if for any two points there is exactly one line containing both of them, we must have $m \geq n$.

(b) Show that any projective plane must have the same number of points and lines.

(c) (difficult) Show that for any prime p, there is a projective plane containing $p^2 + p + 1$ points.

3 Geometry Problems

Problem 3.1. Assume that the points $v_1, ..., v_m \in \mathbb{R}^n$ determine exactly two distances, i.e that there are two numbers a, b such that the distance between any two of the points is either a or b. Consider the polynomials

$$f_i(x_1, ..., x_n) = (||x - v_i||^2 - a^2)(||x - v_i||^2 - b^2)$$

(a) Show that the polynomials $f_1, ..., f_m$ are linearly independent.

(b) Find a set of at most $\binom{n+4}{4}$ polynomials whose span contains all the polynomials $f_1, ..., f_m$. Conclude that $m \leq \binom{n+4}{4}$.

(c) Can you improve the bound by finding a smaller set of polynomials whose span contains all the polynomials $f_1, ..., f_m$? What's the best bound you can get?

(d) Show that there are $\binom{n+1}{2}$ points in $\mathbb{R}^n$ which determine two distances.

(e) (difficult) Can you find a similar bound for the maximum number of points in $\mathbb{R}^n$ determining at most s distances?

Problem 3.2. In this problem we aim to show that there are no 4 points in the plane such that the distance between each pair is an odd integer. Assume for contradiction that there are four such points. We may as well assume that one of them is 0 and that the other ones are a, b and c. Then $||a||, ||b||, ||c||, ||a - b||, ||a - c||, ||b - c||$ are all odd integers.

(a) Show that for any odd integer m, we have that $m^2 \equiv 1 \pmod 8$.

(b) Show that $2\langle a, b\rangle \equiv 1 \pmod 8$.

(c) Let $A = \begin{bmatrix} a & b & c \end{bmatrix}$. Show that $A^T A$ has rank 3, and derive a contradiction from this.

Problem 3.3. (difficult) Show that there cannot be more than $\binom{n+1}{2}$ equiangular lines in $\mathbb{R}^n$. (Recall that if the angle between two vectors v, w in $\mathbb{R}^n$ is θ, we have that $cos\ \theta = \frac{\langle v,w\rangle}{|v||w|}$).

4 Algorithmic Problems

Problem 4.1. (Computing Fibonacci numbers) The Fibonacci numbers are defined by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. So the first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13$.

(a) Write down the recurrence relation defining the Fibonacci numbers on matrix-vector form.

(b) Given a number x, show that we can compute the value of x^{2^k} by performing k multiplications.

(c) Given a number x, show that we can compute the value of x^n by performing at most $2\ log_2(n)$ multiplications.

(d) Devise an algorithm for computing the n^{th} Fibonacci number by performing only $O(log_2(n))$ multiplications and additions.

Problem 4.2. Given a graph with n vertices, we might want to be able to find out whether or not there exists a triangle in the graph (i.e three vertices all of which are connected by edges). Assume that we have an algorithm computing the product of two matrices using at most $O(n^\alpha)$ operations, for some constant α.

(a) Show that there is an algorithm that uses at most $O(n^\alpha)$ operations which, given a graph with n vertices, determines whether or not that graph has a triangle.

(b) Show that there is an algorithm that uses at most $O(n^{k\alpha})$ operations which, given a graph with n vertices, determines whether or not that graph has a subset of $3k$ vertices all of which are connected by edges.

Problem 4.3. Given two graphs G and H we might be interested in determining whether or not they are isomorphic - that is whether there exists a permutation of the vertices of G turning it into H. What is wrong with the following argument that there is a polynomial algorithm for deciding whether or not two graphs are isomorphic? Without loss of generality the vertex sets of G and H are both $\{1, 2, ..., n\}$. We first calculate the eigenvalues and eigenvectors of the adjacency matrices of G and H. It is then easy to test whether there is a permutation of $\{1, 2, ..., n\}$ that (when applied to the coordinates) takes each eigenvector of G to an eigenvector of H with the same eigenvalue, and G and H are isomorphic if and only if such a permutation exists.

Problem 4.4. (difficult) Multiplying two matrices A, B takes a lot of time. But what if you already have a candidate matrix C, and want to check if $AB = C$? Assume you are happy to accept a small probability of error. Devise an algorithm which given three matrices A, B, C checks if $AB = C$ in $O(n^2)$, getting the correct answer with probability $> 99\%$.

5 Graph Problems

Problem 5.1. Let G be a graph with adjacency matrix A. Show that the $(i, j)-$entry of A^k counts the number of walks of length k from i to j in G.

Problem 5.2. Calculate the eigenvalues of K_n and $K_{m,n}$. (K_n is the complete graph on n vertices, ie the graph for which all pairs of vertices are connected by an edge. $K_{m,n}$ is the complete bipartite graph with n vertices in the first vertex class and m in the second - look up what this means!)

Problem 5.3. Given is a graph G on n vertices with adjacency matrix A whose eigenvalues are $\lambda_1 \geq ... \geq \lambda_n$. Let Δ denote the largest degree of any vertex in G, and let δ denote the smallest degree. Show that:

(a) $|\lambda_i| \leq \Delta$ for every i.

(b) If G is connected, then G is regular if and only if $\lambda_1 = \Delta$.

(c) If G is connected, then G is regular and bipartite if and only if $\lambda_n = -\Delta$. (Look up what a bipartite graph is!)

(d) $\delta \leq \lambda_1 \leq \Delta$.

Problem 5.4. Prove that the matrix J (all of whose entries are 1) is a polynomial in the adjacency matrix of a graph G if and only if G is regular and connected.

Problem 5.5. (difficult) Let G be a graph in which every edge is in a unique triangle and every non-edge is a diagonal of a unique $4-$cycle. Show that G is $k-$regular, for some k, and that the number of vertices of G is $k^2/2+1$. Show also that $k \in \{2, 4, 14, 22, 112, 994\}$.

Problem 5.6. (difficult) Show that the edges of K_{10} can't be partitioned into three disjoint Petersen graphs.

6 Hamel Basis Problems

Problem 6.1. Are there any functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, but which are not on the form $f(x) = ax$ for some constant $a \in \mathbb{R}$. [Hint: Look up what a Hamel basis is.]

7 Challenge Problems

Problem 7.1. There are 13 coins with real numbers on them, such that given any coin, the other 12 can be split into two groups of 6 each, such that the sum of numbers of one group is the same as the sum of numbers of the other. Show that all the coins have the same numbers.

Problem 7.2. Find all real numbers r such that a $1 \times r$ rectangle can be tiled by (finitely many) squares

Problem 7.3. You are at a party where no-one has more friends than you do. You discover that every two people there have exactly one mutual friend present. Prove that everybody is your friend.

8 Hints

Problem 1.1.

(a) Using Gaussian elimination we can put the matrix on diagonal form. Why does this preserve the row and column ranks?

(b) Use part (a) to conclude that it's enough to show $rk(AB) \leq rk(B)$. Then think of the matrices as linear maps between vector spaces. How does the rank of a matrix relate to the image of the corresponding map?

Problem 1.2. (a) Use the second method in the Oddtown proof from the lecture.

Problem 1.3. Use the second method in the Oddtown proof from the lecture.

Problem 1.4. Use the second method in the Oddtown proof from the lecture. It might be useful to show that the matrix $M^T M$ is positive definite.

Problem 1.6. Let U be the vector space over $\mathbb{F}_2$ which is spanned by the indicator vectors of the clubs, then consider $U^{\perp}$.

Problem 2.1. (a) Think of the lines as sets of points, and consider the indicator vector v_i of each line l_i. Can you show that the vectors $v_1, ..., v_m$ span the entire space?

Problem 3.1. (a) What happens if you evaluate f_i at v_j?

Problem 3.2. (c) Compute the determinant of $A^T A$ modulo 8 and show that it's non-zero.

Problem 5.3.

(b) Consider the vector $(1, 1, ..., 1)$.

(d) How does λ_1 relate to the value of $x^T Ax$ for a vector x?

Problem 7.1. Start by solving the problem when all the numbers on the coins are integers.