BOOLEAN ELEMENTS IN THE BRUHAT ORDER

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Abstract. We show that $w \in W$ is boolean if and only if it avoids a set of Billey-Postnikov patterns, which we describe explicitly. Our proof is based on an analysis of inversion sets, and it is in large part type-uniform. We also introduce the notion of linear pattern avoidance, and show that boolean elements are characterized by avoiding just the 3 linear patterns $s_1s_2s_1 \in W(A_2)$, $s_2s_1s_3s_2 \in W(A_3)$, and $s_2s_1s_3s_4s_2 \in W(D_4)$.

We also consider the more general case of k-boolean Weyl group elements. We say that $w \in W$ is k-boolean if every reduced expression for w contains at most k copies of each generator. We show that the 2-boolean elements of the symmetric group S_n are characterized by avoiding the patterns $3421, 4312, 4321$, and 456123, and give a rational generating function for the number of 2-boolean elements of S_n.

1. Introduction

Billey and Postnikov [1] defined a notion of pattern avoidance in Weyl groups, which efficiently characterizes those Weyl group elements w whose corresponding Schubert variety X_w is (rationally) smooth, for arbitrary Weyl groups, generalizing the well-known result of Lakshmibai and Sandhya [7] that says for a permutation w, its Schubert variety X_w is smooth if and only if w avoids 3412 and 4231. Since then, Billey-Postnikov patterns (BP patterns), besides geometric importance, have seen many combinatorial applications as well, characterizing fully commutative elements [1], chromobruhatic elements [11], separable elements [3, 4], and so on.

In this paper, we showcase another combinatorial application of BP patterns (Definition 2.3), by characterizing *boolean* elements of arbitrary Weyl groups, generalizing a result by Tenner [9] for the symmetric group, who showed that a permutation w is boolean if and only if it avoids 321 and 3412. Let Φ be any finite crystallographic root system with Weyl group $W = W(\Phi)$ (see more background in Section 2).

Definition 1.1. An element $w \in W$ is called *boolean* if the interval $[\mathrm{id}, w]$ in the (strong) Bruhat order is isomorphic to a Boolean lattice.

Here is the first version of our main theorem.

Theorem 1.2. *Let Φ be a root system. An element $w \in W(\Phi)$ is boolean if and only if w avoids all the BP patterns in Table 1.*

See Table 2 for labels on the Dynkin diagram, where we use s_i to denote the reflection across the simple root α_i. We omit root systems of rank 2 since no confusion will arise.

Theorem 1.2 is notable because in [9], Tenner showed that an element being boolean is equivalent to avoiding 10 patterns in type B and avoiding 20 patterns in

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type	*forbidden patterns*	*# patterns*
A_2	$s_1s_2s_1 = s_2s_1s_2$ *(321)*	*1*
A_3	$s_2s_1s_3s_2$ *(3412)*	*1*
$B_2 = C_2$	$s_1s_2s_1,\ s_2s_1s_2,\ s_1s_2s_1s_2 = s_2s_1s_2s_1$	*3*
B_3	$s_2s_1s_3s_2$	*1*
C_3	$s_2s_1s_3s_2$	*1*
D_4	$s_2s_1s_3s_4s_2$	*1*
G_2	*all patterns of Coxeter length at least 3*	*7*

TABLE 1. Forbidden patterns for boolean elements in Weyl groups

type D, with a certain notion of pattern avoidance for signed permutations, while we only need 7 BP patterns in type B and 3 BP patterns in type D.

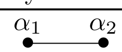
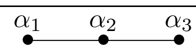
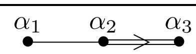
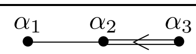
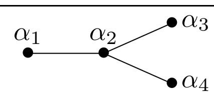
type	Dynkin diagram	pattern π	inversions $I_\Phi(\pi)$
A_2		$s_1s_2s_1$	$\{\alpha_1, \alpha_2, \alpha_1{+}\alpha_2\}$
A_3		$s_2s_1s_3s_2$	$\{\alpha_2, \alpha_1{+}\alpha_2, \alpha_2{+}\alpha_3, \alpha_1{+}\alpha_2{+}\alpha_3\}$
B_3		$s_2s_1s_3s_2$	$\{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$
C_3		$s_2s_1s_3s_2$	$\{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3\}$
D_4		$s_2s_1s_3s_4s_2$	$\{\alpha_2, \alpha_1{+}\alpha_2, \alpha_2{+}\alpha_3, \alpha_2{+}\alpha_4, \alpha_1{+}2\alpha_2{+}\alpha_3{+}\alpha_4, \}$

TABLE 2. Patterns of interest and their inversions

Moreover, we also introduce a new notion of *linear patterns* (Definition 2.4), which simultaneously generalizes the classical folding of root systems and root system embedding [1]. This notion allows us to derive an even simpler characterization of boolean elements, which requires only the same 3 patterns in all types. The following is the second version of our main theorem.

Theorem 1.3. *Let Φ be an irreducible root system. An element $w \in W(\Phi)$ is boolean if and only if w avoids the linear patterns $s_1s_2s_1 \in W(A_2)$, $s_2s_1s_3s_2 \in W(A_3)$, and $s_2s_1s_3s_4s_2 \in W(D_4)$.*

The paper is organized as follows. In Section 2, we provide necessary background and definitions on Weyl groups and pattern avoidance. In Section 3, we prove the two versions of our main theorems by first proving Theorem 1.3 and then deriving Theorem 1.2 from Theorem 1.3. Our proof is largely type-uniform and is completely independent of that of Tenner [9, 10], even in the case of type A root systems whose Weyl group is isomorphic to the symmetric group. Finally in Section 4, we go back to the symmetric group and generalize the notion of boolean permutations to k-boolean permutations, characterize 2-boolean permutations by pattern avoidance (as the case $k \geq 3$ does not seem to be governed by pattern avoidance), and enumerate them.

2. Background on Weyl groups and patterns

We refer readers to [6] for a detailed treatment on root systems.

Throughout the paper, let $\Phi \subset E$ be a finite crystallographic root system of rank r inside an Euclidean space $E \simeq \mathbb{R}^r$ with a positive definite symmetric bilinear form $\langle -, - \rangle$. We fix a choice of positive roots $\Phi^+ \subset \Phi$ which corresponds to a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$. Let $W = W(\Phi)$ be its Weyl group, which is a finite subgroup of $\mathrm{GL}(E)$ generated by reflections $s_\alpha \in \mathrm{GL}(E)$ for all roots α, or equivalent, by s_α's for $\alpha \in \Delta$. For simplicity of notations, we write s_i for s_{α_i} where $\alpha_i \in \Delta$ and we call these reflections *simple reflections*.

The *(strong) Bruhat order* on W, which naturally comes from the Bruhat decomposition of the flag variety, is defined to be the transitive closure of $w \lessdot ws_\beta$ if $\ell(w) = \ell(ws_\beta) - 1$, where ℓ denotes the Coxeter length. There is a minimum id and a maximum w_0 of the Bruhat order. The Bruhat order satisfies the *subword property*, that says if $v < u \in W$ and $u = s_{i_1} \cdots s_{i_\ell}$ is a reduced expression, then there exists a subword of $s_{i_1} \cdots s_{i_\ell}$ that is a reduced expression for v.

A root system Φ is *irreducible* if it cannot be properly partitioned into $\Phi_1 \sqcup \Phi_2$ such that $\langle \beta_1, \beta_2 \rangle = 0$ for all $\beta_1 \in \Phi_1$ and $\beta_2 \in \Phi_2$. Irreducible root systems can be completely classified into 4 infinite families A_n, B_n, C_n, D_n and exceptional types E_6, E_7, E_8, F_4, G_2. We adopt the following conventions for the classical types, as in [6]:

- type A_{n-1}: $\Phi = \{e_i - e_j \,|\, 1 \le i, j \le n\} \subset \mathbb{R}^n/(1, \ldots, 1)$, $\Phi^+ = \{e_i - e_j \,|\, 1 \le i < j \le n\}$, $\Delta = \{e_i - e_{i+1} \,|\, 1 \le i \le n-1\}$;
- type B_n: $\Phi = \{\pm e_i \pm e_j | 1 \le i < j \le n\} \cup \{\pm e_i | 1 \le i \le n\}$, $\Phi^+ = \{e_i \pm e_j | 1 \le i < j \le n\} \cup \{e_i \,|\, 1 \le i \le n\}$, $\Delta = \{e_i - e_{i+1} \,|\, 1 \le i \le n-1\} \cup \{e_n\}$;
- type C_n: $\Phi = \{\pm e_i \pm e_j \,|\, 1 \le i < j \le n\} \cup \{\pm 2e_i \,|\, 1 \le i \le n\}$, $\Phi^+ = \{e_i \pm e_j \,|\, 1 \le i < j \le n\} \cup \{2e_i \,|\, 1 \le i \le n\}$, $\Delta = \{e_i - e_{i+1} \,|\, 1 \le i \le n-1\} \cup \{2e_n\}$;
- type D_n: $\Phi = \{\pm e_i \pm e_j \,|\, 1 \le i < j \le n\}$, $\Phi^+ = \{e_i \pm e_j \,|\, 1 \le i < j \le n\}$, $\Delta = \{e_i - e_{i+1} \,|\, 1 \le i \le n-1\} \cup \{e_{n-1} + e_n\}$.

Note that the root system of type B_2 is isomorphic to C_2. And when we talk about root system of type D_n, we assume $n \ge 4$ as D_3 is the same as A_3.

The *root poset* is the partial order on Φ^+ such that $\alpha \le \beta \in \Phi^+$ if $\beta - \alpha$ can be written as a nonnegative (integral) linear combination of simple roots. The minimal elements of the root poset are precisely the simple roots Δ and there exists a unique maximum of the root poset called the *highest root*. The root poset can be given the structure of a graded poset with the rank of a root being the sum of coefficients of this root in the simple root basis, known as the *height* of this root. We say that a positive root β is *supported on* a simple root $\alpha \in \Delta$ if $\beta \ge \alpha$ in the root poset. Define the *support* of β to be

$$\mathrm{Supp}(\beta) := \{\alpha \in \Delta \,|\, \beta \text{ is supported on } \alpha\} \subset \Delta.$$

For $w \in W(\Phi)$, its *inversion set* is

$$I_\Phi(w) = \{\beta \in \Phi^+ \,|\, w\beta \in \Phi^-\}.$$

We say that β is an *inversion* of w if $\beta \in I_\Phi(w)$, and a *(right) descent* of w if $\beta \in I_\Phi(w) \cap \Delta$ is an inversion and also a simple root. It is a standard fact that $\ell(w) = |I_\Phi(w)|$. The following lemma follows from definitions, with proof omitted.

Lemma 2.1. *Let $w \in W(\Phi)$ and $\alpha \in \Delta$ such that $\ell(ws_\alpha) = \ell(w) + 1$. Then*

$$I_\Phi(ws_\alpha) = s_\alpha I_\Phi(w) \cup \{\alpha\}.$$

The next proposition is useful and well-known (see for example [5]).

Proposition 2.2. *The inversion set uniquely characterizes a Weyl group element. In other words, $I_\Phi : W \to 2^{\Phi^+}$ is injective. Moreover, a subset $I \subset \Phi^+$ is the inversion set of some Weyl group elements if and only if it is biconvex; that is, if and only if:*

(1) *if $\alpha, \beta \in I$, $\alpha + \beta \in \Phi^+$, then $\alpha + \beta \in I$ and,*
(2) *if $\alpha, \beta \notin I$, $\alpha + \beta \in \Phi^+$, then $\alpha + \beta \notin I$.*

We can now introduce a restriction map, defined by Billey and Postnikov [1]. Let $E' \subset E$ be a subspace and $\Phi' = \Phi \cap E'$ is then a root system with an inherited set of positive roots $(\Phi')^+ = \Phi^+ \cap E'$. For any $w \in W(\Phi)$, its inversion set $I_\Phi(w)$ is biconvex and it is easy to see that the restriction $I_\Phi(w) \cap E'$ is also biconvex. By Proposition 2.2, there is a unique element $w' \in W(\Phi')$ such that $I_{\Phi'}(w') = I_\Phi(w) \cap E'$. We call such w' the *restriction* of w to Φ', denoted $w|_{\Phi'}$.

Definition 2.3. We say that $w \in W(\Phi)$ contains the BP (Billey-Postnikov) pattern $\pi \in W(R)$, where choices of positive roots $\Phi^+ \subset \Phi$ and $R^+ \subset R$ have been fixed, if there exists a subspace $E' \subset E$ such that there is an isomorphism between root systems $\Phi' := \Phi \cap E$ and R that preserves the chosen positive roots and maps $w|_{\Phi'}$ to π.

We also introduce a new notion of *linear patterns*, which enables an even nicer characterization of boolean elements.

Definition 2.4. We say that $w \in W(\Phi)$ contains the *linear pattern* $\pi \in W(R)$, where choices of positive roots $\Phi^+ \subset \Phi$ and $R^+ \subset R$ have been fixed, if there exists a linear transformation $R \to \Phi$ that maps positive roots R^+ to positive roots Φ^+, inversions $I_R(\pi)$ of π to inversions $I_\Phi(w)$ of w, and non-inversions $R^+ \setminus I_R(\pi)$ to non-inversions $\Phi^+ \setminus I_\Phi(w)$. If the simple roots $\alpha_1, \ldots, \alpha_k$ of R are mapped to $\beta_1, \ldots, \beta_k$, then we say that w contains π *generated at* $\beta_1, \ldots, \beta_k$.

We note that if w contains the BP pattern π, then w also contains the linear pattern π, but not necessarily the other way around. The difference between linear pattern containment and BP pattern containment is that in linear patterns, we do not require the map to be injective or angle-preserving, and we are also not required to map to all vectors in a subspace (we might map to only a strict subset of the vectors in a subspace). For example, there are linear patterns $\pi \in W(A_7)$ in $w \in W(E_7)$, and $\pi \in W(A_2)$ in $w \in W(B_2)$, but this is not the case for BP patterns. We proceed to give an example that demonstrates what linear patterns can look like.

Example 2.5. Let α_1 be the long simple root of B_2 and α_2 be the short simple root of B_2. Then $s_1s_2s_1 \in W(B_2)$ contains the linear pattern $s_2s_1s_3s_2 \in W(A_3)$. Letting the simple roots of A_3 be $\beta_1, \beta_2, \beta_3$, this is demonstrated by sending $\beta_1 \mapsto \alpha_2$, $\beta_2 \mapsto \alpha_1$, $\beta_3 \mapsto \alpha_2$. The rest of the map is then uniquely defined by linearity. As $I_{A_3}(s_2s_1s_3s_2) = \{\beta_2, \beta_1+\beta_2, \beta_2+\beta_3, \beta_1+\beta_2+\beta_3\}$, $(A_3)^+ \setminus I_{A_3}(s_2s_1s_3s_2) = \{\beta_1, \beta_3\}$, $I_{B_2}(s_1s_2s_1) = \{\alpha_1, \alpha_1+\alpha_2, \alpha_1+2\alpha_2\}$, $(B_2)^+ \setminus I_{B_2}(s_1s_2s_1) = \{\alpha_2\}$, we then see that inversions are sent to inversions, and non-inversions are sent to non-inversions. See Figure 1.

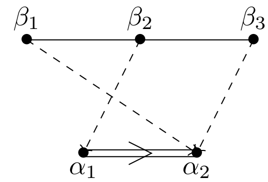


FIGURE 1. The linear pattern $s_2s_1s_3s_2 \in W(A_3)$ in $s_1s_2s_1 \in W(B_2)$

To further illustrate how BP patterns and linear patterns compare, we note that for π in a type A_k or D_k Weyl group and w in a type A_n or D_n Weyl group, w contains the linear pattern π iff w contains the BP pattern π. We will not use this fact anywhere in the paper, and we only give a brief sketch of the proof. One can start by noting that in the case where both R and Φ are irreducible and simply laced, the linear map in fact preserves angles, and further that for A_n and D_n, it turns out that the linear pattern hits all the roots in the $\mathbb{R}$-span of the image, since there are no $A_n \subset D_n$ or $D_n \subset A_n$.

3. Proof of the main theorem

We begin with the following simple proposition.

Proposition 3.1. *An element $w \in W$ is boolean if and only if any reduced expression (or equivalently, all reduced expressions) of w does not contain repeated letters.*

A special case of Proposition 3.1 appears as Proposition 7.3 in [9] in the case of finite classical types, with the proof omitted.

Proof. If w is boolean, then the interval $[\mathrm{id}, w]$ has the same number of atoms as the height. The atoms of $[\mathrm{id}, w]$ are the simple reflections used by any reduced expression of w while the height is $\ell(w)$. This implies that any reduced expression cannot contain repeated letters. Conversely, if w is a product of distinct simple reflections, $[\mathrm{id}, w]$ being boolean follows directly from the subword property of the strong Bruhat order. $\square$

We will first prove Theorem 1.3. Then in Section 3.2, we deduce the BP pattern version from Theorem 1.3.

3.1. **Proof of Theorem 1.3.**

We start the proof with a very useful lemma.

Lemma 3.2. *Let $w \in W(\Phi)$ and $\alpha \in \Delta$ be a simple root. Then any (or equivalently, all) reduced expression of w contains s_α if and only if there exists $\beta \in I_\Phi(w)$ supported on α.*

Proof. Use induction on $\ell(w)$. The claim is clearly true when $\ell(w) = 0$, where w is the identity, with one reduced expression being the empty string and $I_\Phi(w) = \emptyset$.

For the general case, assume first that there exists $\beta \in I_\Phi(w)$ supported on α. We want to show that all reduced expressions of w contain s_α. A reduced expression of w must end with $s_{\alpha'}$, where $\alpha' \in \Delta$ is a descent of w. If $\alpha' = \alpha$, we are done. If $\alpha' \neq \alpha$, by Lemma 2.1, since $\beta \in I_\Phi(w)$, $s_{\alpha'}\beta \in I_\Phi(ws_{\alpha'})$. Since $\alpha \neq \alpha'$, $s_{\alpha'}\beta = \beta - \big(2\langle\alpha', \beta\rangle/\langle\alpha', \alpha'\rangle\big)\alpha'$ is also supported on α. By induction hypothesis, all reduced expressions of $ws_{\alpha'}$ contain s_α, so all reduced expressions of w that end with $s_{\alpha'}$ contain s_α. As we know this for every descent α', we know that all reduced expressions of w contain s_α.

For the other direction, let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression and choose the largest k such that $s_{i_k} = s_\alpha$. For $j = 0, 1, \ldots, \ell$, write $w^{(j)} = s_{i_1} \cdots s_{i_j}$ so that $w^{(0)} = \mathrm{id}$ and $w^{(\ell)} = w$. We use induction on j from k to ℓ to show that there exists $\beta_j \in I_\Phi(w^{(j)})$ such that β_j is supported on α. For $j = k$, take $\beta_k = \alpha$ since α is a descent for $w^{(k)}$. Now suppose we have β_j constructed. By Lemma 2.1, since $\beta_j \neq \alpha_{i_{j+1}}$, where $s_{i_{j+1}}$ denotes the reflection across the simple root $\alpha_{i_{j+1}}$, $s_{i_{j+1}}\beta_j \in I_\Phi(w^{(j+1)})$. We have $s_{i_{j+1}}\beta_j = \beta_j - \big(2\langle\alpha_{i_{j+1}}, \beta_j\rangle / \langle\alpha_{i_{j+1}}, \alpha_{i_{j+1}}\rangle\big)\alpha_{i_{j+1}}$ is supported on α, since β_j does and $\alpha_{i_{j+1}} \neq \alpha$ by maximality of k. Pick $\beta_{j+1} = s_{i_{j+1}}\beta_j$ and the induction step goes through. In the end, we conclude that there exists $\beta_\ell \in I_\Phi(w)$ that is supported on α as desired. □

Remark 3.3. In the case of type A_{n-1} where the Weyl group $W(A_{n-1})$ is isomorphic to the symmetric group $\mathfrak{S}_n$, Lemma 3.2 is saying that the simple transposition $s_k = (k\ k+1)$ appears in a reduced expression of w if there exists i, j such that $i \leq k < j$ and $w(i) > w(j)$. This fact can be easily observed.

The following technical lemma, which is purely root-theoretic, is going to be important. It is also the only part of the proof that is not type-uniform.

Lemma 3.4. *Let $\alpha \in \Delta$ be a simple root and $\beta \neq \alpha \in \Phi^+$ be a positive root such that $s_\alpha\beta \in \Phi^+$ is supported on α. Then (at least) one of the following is true:*

1. *$\beta + \alpha \in \Phi^+$;*
2. *$\beta = \alpha + \gamma_1 + \gamma_2$ such that $\alpha + \gamma_1, \alpha + \gamma_2 \in \Phi^+$ for some $\gamma_1, \gamma_2 \in \Phi^+$;*
3. *$\beta = 2\alpha + \gamma_1 + \gamma_2 + \gamma_3$ such that $\alpha + \gamma_i \in \Phi^+$ for $i \in \{1, 2, 3\}$, $\alpha + \gamma_i + \gamma_j \in \Phi^+$ for $i \neq j \in \{1, 2, 3\}$, and $\beta - \alpha \in \Phi^+$ for some $\gamma_1, \gamma_2, \gamma_3 \in \Phi^+$.*

Proof. Let us first reduce to the case where Φ is irreducible. We split $\beta = \beta_1 + \beta_2$, where β_1 is the projection of β to the span of the irreducible component containing the simple root α, and β_2 is the projection of β to the orthogonal complement. Note that the assumption of the lemma then holds for the pair α, β_1. Then assuming the lemma in the irreducible case, we get that either (1) $\beta_1 + \alpha \in \Phi^+$, in which case also $\beta + \alpha \in \Phi^+$; or (2) there is a decomposition $\beta_1 = \alpha + \gamma_1 + \gamma_2$, in which case we can also decompose $\beta = \alpha + \gamma_1 + (\gamma_2 + \beta_2)$; or (3) there is a decomposition $\beta_1 = 2\alpha + \gamma_1 + \gamma_2 + \gamma_3$, in which case we can also decompose $\beta = 2\alpha + \gamma_1 + \gamma_2 + (\gamma_3 + \beta_2)$. So it remains to prove the lemma for an irreducible root system.

For the classical types, we carry out a manual case check on the standard constructions. We will proceed type by type, starting from the simply laced types.

Type A_n: α is a simple root $e_i - e_{i+1}$, and β is a positive root $e_j - e_k$ for some $j < k$. Keeping in mind that $s_\alpha(\beta)$ is supported on α, there are a few options:

1. $j < i < i+1 < k$. Then we decompose $\beta = (e_i - e_{i+1}) + (e_j - e_i) + (e_{i+1} - e_k)$, as in (2).
2. $j = i+1$. Then $\beta + \alpha \in \Phi^+$, as in (1).
3. $k = i$. Then $\beta + \alpha \in \Phi^+$, as in (1).

Type D_n: Due to the automorphism of the Dynkin diagram of D_n, we can assume that $\alpha = e_i - e_{i+1}$. If $\beta = e_j - e_k$, then we are in the type A_{n-1} subsystem, so we are done by the type A_n case (crucially, we use the fact that α is also a simple root of this A_{n-1}, and β is supported on α when taken as a root of A_{n-1}). So this leaves us with the case $\beta = e_j + e_k$ (with $j < k$). We split into a few options for α:

- $\alpha = e_{n-1} - e_n$. Then there are a few options for the indices, keeping in mind that $s_\alpha(\beta)$ is supported on α.
  * $k < n-1$. Then we decompose $\beta = (e_{n-1} - e_n) + (e_j + e_n) + (e_k - e_{n-1})$, as in (2).
  * $j < n-1 < k = n$. Then $\beta + \alpha \in \Phi^+$, as in (1).
- $\alpha = e_i - e_{i+1}$ for $i < n-1$. We again split into cases for the indices.
  * $j < i$. Split into cases again.
    · $k \neq i, i+1$. Then we decompose $\beta = (e_i - e_{i+1}) + (e_j - e_i) + (e_{i+1} + e_k)$, as in (2).
    · $k = i+1$. Then $\beta + \alpha \in \Phi^+$, as in (1).
    · $k = i$. Then we decompose $\beta = 2(e_i - e_{i+1}) + (e_j - e_i) + (e_{i+1} - e_n) + (e_{i+1} + e_n)$, as in (3).
  * $j = i, k = i+1$. Then we decompose $\beta = (e_i - e_{i+1}) + (e_{i+1} - e_n) + (e_{i+1} + e_n)$, as in (2).
  * $j = i+1$. Then $\beta + \alpha \in \Phi^+$, as in (1).

Type B_n: If α, β are both in the type A_{n-1} subsystem, then we are done by the type A_n case, as before. It remains to consider the case where $\alpha = e_n$, $\beta = e_k$, or $\beta = e_j + e_k$ (with $j < k$). We proceed to check these cases.

- $\alpha = e_n$. There are a few options for β, again keeping in mind that $s_\alpha(\beta)$ is supported on α.
  * $\beta = e_k$. Then $\beta + \alpha \in \Phi^+$, as in (1).
  * $\beta = e_j - e_n$. Then $\beta + \alpha \in \Phi^+$, as in (1).
  * $\beta = e_j + e_k$ with $k < n$. Then we decompose $\beta = e_n + (e_k - e_n) + e_j$, as in (2).
- $\beta = e_k$. Then the only case we have not considered yet is $\alpha = e_i - e_{i+1}$. Given that $s_\alpha(\beta)$ is supported on α, there are a few options for the indices.
  * $k < i$. Then we decompose $\beta = (e_i - e_{i+1}) + (e_k - e_i) + e_{i+1}$, as in (2).
  * $k = i+1$. Then we decompose $\beta + \alpha \in \Phi^+$, as in (1).
- $\beta = e_j + e_k$. The remaining case is again $\alpha = e_i - e_{i+1}$. Keeping in mind that $s_\alpha(\beta)$ is supported on α, there are a few options for the indices.
  * $j < i$. We split into cases for k.
    · $k \neq i, i+1$. Then we decompose $\beta = (e_i - e_{i+1}) + (e_j - e_i) + (e_{i+1} + e_k)$, as in (2).
    · $k = i+1$. Then $\beta + \alpha \in \Phi^+$, as in (1).
    · $k = i$. Then we decompose $\beta = 2(e_i - e_{i+1}) + (e_j - e_i) + e_{i+1} + e_{i+1}$, as in (3).
  * $j = i, k = i+1$. Then we decompose $\beta = (e_i - e_{i+1}) + e_{i+1} + e_{i+1}$, as in (2).
  * $j = i+1$. Then $\beta + \alpha \in \Phi^+$, as in (1).

Type C_n: Again, if α, β are both in the type A_{n-1} subsystem, then we are done by the type A_n case. It remains to consider the case where $\alpha = 2e_n$, $\beta = 2e_k$, or $\beta = e_j + e_k$ (with $j < k$).

- $\alpha = 2e_n$. There are a few options for β, again keeping in mind that $s_\alpha(\beta)$ is supported on α.
  - $\beta = 2e_k$. Then we decompose $\beta = 2e_n + (e_k - e_n) + (e_k - e_n)$, as in (2).
  - $\beta = e_j - e_n$. Then $\beta + \alpha \in \Phi^+$, as in (1).
  - $\beta = e_j + e_k$ with $k < n$. Then we decompose $\beta = 2e_n + (e_j - e_n) + (e_k - e_n)$, as in (2).
- $\beta = 2e_k$. The only case that is left is $\alpha = e_i - e_{i+1}$. Given that $s_\alpha(\beta)$ is supported on α, there are a few options for the indices.
  - $k < i$. Then we decompose $\beta = (e_i-e_{i+1})+(e_k-e_i)+(e_k+e_{i+1})$, as in (2).
  - $k = i+1$. Then $\beta + \alpha \in \Phi^+$, as in (1).
- $\beta = e_j + e_k$. The remaining case is again $\alpha = e_i - e_{i+1}$. Note that for C_n, $e_s - e_t$ is supported on $e_i - e_{i+1}$ iff $s \leq i$ and $i + 1 \leq t$, and that $e_s + e_t$ is supported on $e_i - e_{i+1}$ iff $s \leq i$. In fact, this condition is the same for B_n, so the cases for the indices i, j, k that are possible are exactly the same as for the analogous case for B_n. We can even use all the same assignments to options (1), (2), and (3) as for B_n, except for cases where e_s appears as $\alpha+\beta$ in option (1) or in a decomposition (in options (2) or (3)). One can go through the list of cases and confirm that this only happens twice. We now consider these two cases for C_n.
  - $j < i, k = i$. Then we decompose $\beta = (e_i - e_{i+1}) + (e_j - e_i) + (e_{i+1} + e_i)$, as in (2).
  - $j = i, k = i + 1$. Then $\beta + \alpha \in \Phi^+$, as in (1).

For the exceptional types G_2, F_4 and E_8, the lemma is checked on a computer. It is easy to check types G_2 and F_4 by hands but we won't do the tedious case analysis here. The cases of E_6 and E_7 follow from E_8, by identifying these as subsystems of E_8. □

We begin by proving that if w contains one of our bad linear patterns, then it is not boolean. The fact that this proposition works so neatly is one of the main motivations for thinking about this in terms of linear patterns (instead of BP patterns).

Proposition 3.5. *For irreducible Φ, if $w \in W(\Phi)$ contains the linear pattern $s_1s_2s_1 \in W(A_2)$, $s_2s_1s_3s_2 \in W(A_3)$, or $s_2s_1s_3s_4s_2 \in W(D_4)$, then w is not boolean.*

Proof. Say for a contradiction that w contains π, one of these 3 linear patterns, but is nevertheless boolean. We first claim that it suffices to show that for any simple root α which is an inversion of w, ws_α still contains the pattern π. To see this, note that ws_α is still boolean, since there is a reduced expression for w ending in s_α (this is Corollary 1.4.6. in [2]), and then by induction, the identity Weyl group element contains the pattern π, which is a contradiction. Suppose w contains the linear pattern π generated at the positive roots $\beta_1, \ldots, \beta_k$, none of which is equal to α. Then by Lemma 2.1, ws_α contains π generated at the positive roots $s_\alpha\beta_1, \ldots, s_\alpha\beta_k$. So the only way to get rid of π is if $\beta_i = \alpha$ for at least one $i \in [k]$. It remains to do some casework.

- If $\pi = s_1s_2s_1 \in W(A_2)$, let α' be the other root generating the linear pattern we are considering (in addition to α). Then $\alpha + \alpha' \in \Phi^+$, so since

Φ is irreducible, $\langle \alpha, \alpha' \rangle < 0$, so $s_\alpha \alpha' = \alpha' - (2\langle \alpha, \alpha' \rangle / \langle \alpha, \alpha \rangle)\, \alpha$ is supported on α. But $s_\alpha(\alpha')$ is an inversion of ws_α, so by Lemma 3.2, any reduced expression of ws_α contains s_α. However, this implies that there is a reduced expression for w that contains two copies of s_α, which is impossible.

- If $\pi = s_2 s_1 s_3 s_2 \in W(A_3)$, then since α is an inversion, it must be the middle simple root β_2, as the middle simple root is the only simple root which is an inversion of π. In analogy to what we saw before, $\beta_1 + \beta_2 \in \Phi^+$ and $\beta_2 + \beta_3 \in \Phi^+$, so $\langle \beta_2, \beta_1 \rangle$ and $\langle \beta_2, \beta_3 \rangle < 0$, from which we get that

$$s_\alpha(\beta_1 + \beta_2 + \beta_3) = \beta_1 - (2\langle \alpha, \beta_1 \rangle / \langle \alpha, \alpha \rangle)\, \alpha - \alpha + \beta_3 - (2\langle \alpha, \beta_3 \rangle / \langle \alpha, \alpha \rangle)\, \alpha$$

  is supported on α. But it is also an inversion of ws_α, so we get a contradiction as before.
- If $\pi = s_2 s_1 s_3 s_4 s_2 \in W(D_4)$, then since α is an inversion, it must be β_2, since again this is the only simple root which is an inversion of π, and we again reach a contradiction because $s_\alpha(\beta_1 + 2\beta_2 + \beta_3 + \beta_4)$ is an inversion of ws_α which is supported on α.

□

For the direction that w avoids the 3 bad patterns implies that w is boolean, our main strategy is induction on the size of $\bigcup_{\beta \in I_\Phi(w)} \mathrm{Supp}(\beta)$ (the number of simple roots supporting some inversion of w) via the following technical lemma.

Lemma 3.6. *If $w \in W(\Phi)$ avoids the 3 bad patterns $s_1 s_2 s_1 \in W(A_2)$, $s_2 s_1 s_3 s_2 \in W(A_3)$, and $s_2 s_1 s_3 s_4 s_2 \in W(D_4)$, and $\alpha \in I_\Phi(w)$ is a simple root, then $I_\Phi(ws_\alpha)$ contains no roots supported on α and moreover, ws_α does not contain any of these 3 bad patterns.*

Proof of Lemma 3.6. Say for a contradiction that w avoids the 3 bad patterns and $\alpha \in I_\Phi(w)$ is a simple root, but there is a root $\gamma \in I_\Phi(ws_\alpha)$ supported on α. We write $\beta = s_\alpha(\gamma)$, and note that $\gamma = s_\alpha(\beta)$, and that $s_\alpha(\beta) \in I_\Phi(ws_\alpha) \implies \beta \in I_\Phi(w)$ by Lemma 2.1. By Lemma 3.4 applied to these α, β, we are now in one of the following three cases:

(1) $\alpha + \beta \in \Phi^+$. Note that α, β are inversions of w, and biconvexity implies that $\alpha + \beta$ is then also an inversion of w. So w contains an $s_1 s_2 s_1 \in W(A_2)$ generated at α, β.

(2) $\beta = \gamma_1 + \alpha + \gamma_2$ with $\gamma_1 + \alpha, \alpha + \gamma_2 \in \Phi^+$. Then if γ_1 or γ_2 is an inversion of w, w contains an $s_1 s_2 s_1 \in W(A_2)$ at respectively γ_1, α or α, γ_2. If neither is an inversion, then we get that $\gamma_1 + \alpha$ and $\alpha + \gamma_2$ are both inversions, since otherwise we would get a contradiction with biconvexity from $\gamma_1, \alpha + \gamma_2, \gamma_1 + \alpha + \gamma_2$ or $\gamma_1 + \alpha, \gamma_2, \gamma_1 + \alpha + \gamma_2$. We have now determined whether all the relevant roots are inversions or non-inversions of w to conclude that w contains an $s_2 s_1 s_3 s_2 \in W(A_3)$ generated at $\gamma_1, \alpha, \gamma_2$.

(3) $\beta = \gamma_1 + 2\alpha + \gamma_2 + \gamma_3$. Then if γ_1, γ_2, or γ_3 is an inversion of w, w contains a $s_1 s_2 s_1 \in W(A_2)$ at respectively γ_1, α or γ_2, α or γ_3, α. We restrict to the remaining case that $\gamma_1, \gamma_2, \gamma_3$ are all non-inversions. If any of $\gamma_1 + \alpha + \gamma_2, \gamma_1 + \alpha + \gamma_3, \gamma_2 + \alpha + \gamma_3$ is an inversion, then w contains a bad pattern by case (2). So we restrict to the case where these three roots are also non-inversions. Now if $\gamma_1 + \alpha$ is a non-inversion, then we get a contradiction with biconvexity considering $\gamma_1 + \alpha, \gamma_2 + \alpha + \gamma_3$. So $\gamma_1 + \alpha$ and analogously $\gamma_2 + \alpha, \gamma_3 + \alpha$ are inversions. Finally, biconvexity implies

that $(\gamma_1+\alpha+\gamma_2)+\gamma_3$ is not an inversion. We have now determined whether all the relevant roots are inversions or non-inversions of w to conclude that w contains $s_2s_1s_3s_4s_2 \in W(D_4)$.

Given the first part of the lemma, we can now deduce the second part, i.e. that ws_α does not contain any of our 3 bad patterns. Suppose it contains the bad pattern π with simple roots mapping to $\beta_1,\dots,\beta_k \in \Phi^+$ (where $k=2$, $k=3$, or $k=4$ depending on the pattern). If some $\beta_i=\alpha$, then we can note for each of our patterns that there is a root in $I_\Phi(ws_\alpha)$ covering α, which is impossible. So no β_i is α. But then it follows from Lemma 2.1 that $s_\alpha(\beta_1),\dots,s_\alpha(\beta_k)$ generate a pattern π in w, which is a contradiction. So ws_α also avoids the 3 bad patterns. $\square$

From here, the proof of the our linear pattern characterization of boolean elements (Theorem 1.3) is short.

Proof of Theorem 1.3. Proposition 3.5 gives one direction. As for the other direction, i.e. that w which does not contain a bad pattern is boolean, we prove it by inducting on the size of $\bigcup_{\beta\in I_\Phi(w)}\operatorname{Supp}(\beta)$. The base case is trivial. As for the inductive step, we find a simple root $\alpha \in I_\Phi(w)$ (which exists e.g. by biconvexity), and consider ws_α. By Lemma 3.6, $\alpha \notin \bigcup_{\beta\in I_\Phi(ws_\alpha)}\operatorname{Supp}(\beta)$. As multiplying a root by s_α only changes the coefficient of α, all other simple roots in $\bigcup_{\beta\in I_\Phi(w)}\operatorname{Supp}(\beta)$ are also in $\bigcup_{\beta\in I_\Phi(ws_\alpha)}\operatorname{Supp}(\beta)$. Putting these observations together, we get that

$$\left|\bigcup_{\beta\in I_\Phi(ws_\alpha)}\operatorname{Supp}(\beta)\right| = \left|\bigcup_{\beta\in I_\Phi(w)}\operatorname{Supp}(\beta)\right| - 1.$$

By Lemma 3.6, ws_α also avoids the bad patterns. So by our inductive hypothesis, ws_α is boolean. We pick a reduced word for ws_α. By Lemma 3.2, this reduced word does not contain s_α. So if we add s_α to the end of this word, we get a word for w in which each generator appear at most once. Since s_α is an inversion of w, this word is reduced. So w is also boolean. This completes the induction. $\square$

3.2. **From linear patterns to BP patterns.** In this section, we deduce the characterization of boolean elements in terms of BP pattern avoidance (Theorem 1.2) from the characterization in terms of linear pattern avoidance (Theorem 1.3). In the next lemma, we show that containing a linear pattern is equivalent to containing a corresponding set of BP patterns (each itself containing this linear pattern). After this lemma, most of this section is some casework on a finite number of Weyl groups to figure out the explicit sets of BP patterns that correspond to our linear patterns.

Lemma 3.7. *Let Φ and R be irreducible root systems, let $w \in W(\Phi)$ and $\pi \in W(R)$, and let k be the rank of R. Then w contains the linear pattern π if and only if w contains at least one BP pattern in the set*

$$P_\pi := \{\sigma \in \bigcup_{\substack{\Theta \text{ irreducible} \\ \operatorname{rank}(\Theta)\le k}} W(\Theta) \colon \sigma \text{ contains the linear pattern } \pi\}.$$

Proof. For the forward implication, restrict to the $\mathbb{R}$-span of the image of R in Φ. Denoting this subspace root system by Θ, we define $\sigma = w|_\Theta$. Note that Θ has rank at most k, Θ is irreducible (this follows by considering the images of simple roots α_i of R and deducing from $\alpha_i + \alpha_j \in \Phi^+ \implies \langle \alpha_i, \alpha_j \rangle < 0$ that they must all lie in the same irreducible component) and σ contains the linear pattern π, so $\sigma \in P_\pi$. Hence, w contains some BP pattern in P_π.

For the backward implication, suppose w contains the BP pattern $\sigma \in P_\pi$, $\sigma \in W(\Theta)$. Then there is a linear map $R \to \Theta$ demonstrating that σ contains the linear pattern π. Composing with the inclusion $\Theta \to \Phi$, we see that π is also a linear pattern of w. $\square$

Firstly, observe that P_π is finite since there are only finitely many irreducible root systems of rank at most k, each having a finite Weyl group. Secondly, observe that it follows from the lemma that avoiding the linear patterns $\pi_1, \ldots, \pi_m$ is equivalent to avoiding all BP patterns in $P_{\pi_1} \cup \ldots \cup P_{\pi_m}$. Finally, observe that if there are $\sigma_1, \sigma_2 \in P$ with σ_1 being a BP pattern in σ_2, then w containing a BP pattern in P is equivalent to w containing a BP pattern in $P \setminus \{\sigma_2\}$. In other words, we can get rid of redundant elements. To make this precise, for any set P or Weyl group elements, we define the *reduction of* P, denoted $\mathrm{red}(P)$, as:

$$\mathrm{red}(P) = \{w \in P \colon w \text{ does not contain any BP pattern } \pi \neq w, \pi \in P\}.$$

With this notation, our observation is that avoiding all BP patterns in P is equivalent to avoiding all BP patterns in $\mathrm{red}(P)$.

For any two sets P, S of Weyl group elements (which we think of as BP patterns), we also define the *reduction of* P *mod* S, denoted P/S, is the set of elements of P which do not contain a BP pattern in S, i.e.

$$P/S := \{w \in P \colon w \text{ does not contain any BP pattern } \pi \in S\}.$$

We now present $P_1 := P_{\pi_1}$ where $\pi_1 = s_1 s_2 s_1 \in W(A_2)$. We find this by just checking all elements of Weyl groups of irreducible root systems of rank at most 2.

Lemma 3.8. *The set of BP patterns corresponding to* π_1*,* $P_1 := P_{\pi_1}$*, consists of*

- $s_1 s_2 s_1 \in W(A_2)$*;*
- $s_2 s_1 s_2, s_1 s_2 s_1 s_2 \in W(B_2)$*;*
- $s_2 s_1 s_2, s_1 s_2 s_1 s_2, s_2 s_1 s_2 s_1, s_1 s_2 s_1 s_2 s_1, s_2 s_1 s_2 s_1 s_2, s_1 s_2 s_1 s_2 s_1 s_2 \in W(G_2)$*.*

Proof. Let us begin by observing that the linear map demonstrating containment of π_1 cannot send both simple roots to the same image, as $\beta \in \Phi^+ \implies \beta + \beta = 2\beta \notin \Phi^+$.

We go through all the irreducible root systems of rank at most 2. There are 4 of these: A_1, A_2, B_2, G_2. By the observation above, there are no $w \in W(A_1)$ containing π_1.

For A_2, suppose the linear map demonstrating containment of π_1 takes the simple roots to β_1, β_2. Then by the observation above, the only option is that β_1, β_2 are the two simple roots of A_2. This determines the inversions as well (namely, $\beta_1, \beta_2, \beta_1 + \beta_2$ are all inversions). The only $w \in W(A_2)$ with these inversions is $w = s_1 s_2 s_1$, which indeed contains π_1 (itself) as a pattern.

For B_2, let us call the simple roots α_1 and α_2. $W(B_2)$ has 8 elements. 5 of these have strictly fewer than 3 inversions (these are id s_1, s_2, $s_1 s_2$, and $s_2 s_1$), and we can immediately conclude that these do not contain a linear π_1, since the observation above implies that $\beta_1, \beta_2, \beta_1 + \beta_2$ are all distinct inversions. We check the 3 remaining $w \in W(B_2)$:

- $s_1s_2s_1 \in W(B_2)$ has the three inversions α_1, $\alpha_1+\alpha_2$, $\alpha_1+2\alpha_2$. No two of these add up to an inversion, so this does not contain a linear π_1.
- $s_2s_1s_2 \in W(B_2)$ has the three inversions $\alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2$. It contains a linear π_1 generated at $\alpha_2, \alpha_1+\alpha_2$.
- $s_1s_2s_1s_2 \in W(B_2)$ has all positive roots as inversions. It contains a linear π_1 generated at α_1, α_2.

This leaves us with G_2. Let us say that the simple roots of G_2 are α_1, α_2. $W(G_2)$ has 12 elements, out of which 5 we can rule out immediately on account of having strictly fewer than 3 inversions – these are $\mathrm{id}, s_1, s_2, s_1s_2$, and s_2s_1. Let us go through the rest:

- $s_1s_2s_1$ has the 3 inversions $2\alpha_1+3\alpha_2$, $\alpha_1+\alpha_2$, α_1. No two of these add up to an inversion, so this does not contain a linear π_1.
- $s_2s_1s_2$ has the 3 inversions $\alpha_1+2\alpha_2$, $\alpha_1+3\alpha_2$, α_2. It contains a linear π_1 generated at $\alpha_1+2\alpha_2, \alpha_2$.
- $s_1s_2s_1s_2$ has the 4 inversions $2\alpha_1+3\alpha_2$, $\alpha_1+2\alpha_2$, $\alpha_1+3\alpha_2$, α_2. It contains a linear π_1 generated at $\alpha_1+2\alpha_2, \alpha_2$.
- $s_2s_1s_2s_1$ has the 4 inversions $\alpha_1+2\alpha_2$, $2\alpha_1+3\alpha_2$, $\alpha_1+\alpha_2$, α_1. It contains a linear π_1 generated at $\alpha_1+2\alpha_2, \alpha_1+\alpha_2$.
- $s_1s_2s_1s_2s_1$ has all positive roots other than α_2 as inversions. It contains a linear π_1 generated at $\alpha_1+2\alpha_2, \alpha_1+\alpha_2$.
- $s_2s_1s_2s_1s_2$ has all positive roots other than α_1 as inversions. It contains a linear π_1 generated at $\alpha_1+2\alpha_2, \alpha_1+\alpha_2$.
- $s_1s_2s_1s_2s_1s_2$ has all positive roots as inversions. It contains a linear π_1 generated at α_1, α_2.

This completes the casework.

□

We proceed to present $P_2 := \mathrm{red}(P_{\pi_2})/P_{\pi_1}$ where $\pi_2 = s_2s_1s_3s_2 \in W(A_3)$.

Lemma 3.9. *The set of additional BP patterns corresponding to π_2, $P_2 := red(P_{\pi_2})/P_{\pi_1}$, consists of*

- $s_1s_2s_1 \in W(B_2)$*;*
- $s_2s_1s_3s_2 \in W(A_3)$*;*
- $s_2s_1s_3s_2 \in W(C_3)$*.*

Proof. $\mathrm{red}(P_{\pi_2})/P_{\pi_1}$ consists of all elements of Weyl groups of root systems of rank at most 3 that contain π_2 and do not contain the linear pattern π_1 nor any smaller BP pattern that contains the linear pattern π_2. The root systems of rank at most 3 are $A_1, A_2, B_2, G_2, A_3, B_3, C_3$. Suppose that a linear map demonstrating containment of π_2 sends the simple roots to $\beta_1, \beta_2, \beta_3$. We note that then $\beta_1+\beta_2+\beta_3$ has to be a root. This already implies that a linear π_2 is not contained in any element of $W(A_1)$ or $W(A_2)$. Also note that $\beta_2, \beta_2+\beta_3$, and $\beta_1+\beta_2+\beta_3$ are all distinct inversions, so if w contains π_2, then w must have at least 3 inversions, and that these inversions are all $\geq \beta_2$ in the root poset (this last fact will be useful later). We can use this inversion count to check the case of B_2. Consider the argument for B_2 in the proof of Lemma 3.8. We note that the same elements are ruled out on account of not having enough inversions. The only element which has not been ruled out and also does not contain π_1 is then $s_1s_2s_1 \in W(B_2)$. It has the inversions $\alpha_1, \alpha_1+\alpha_2, \alpha_1+2\alpha_2$ (here and later, we are letting α_i be the simple roots of the root system into which we are considering a linear map, ordered according to our conventions), so it contains π_2 generated at $\alpha_2, \alpha_1, \alpha_2$. There is no strict subspace which contains a linear π_2, so we conclude that $s_1 s_2 s_1 \in \mathrm{red}(P_{\pi_2})/P_{\pi_1}$

For G_2, the only element left to consider after the proof of Lemma 3.8 and counting inversions is $s_1 s_2 s_1 \in W(G_2)$, which has the inversions $2\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_1$. If $\beta_2 \neq \alpha_1$ in this case, then it is not possible for there to be three distinct inversions containing β_2. Hence, we just need to consider the case that $\beta_2 = \alpha_1$. Arguing similarly, we get that $\beta_2 + \beta_3 = \alpha_1 + \alpha_2 \implies \beta_3 = \alpha_2$. Continuing, $\beta_1 + \beta_2 + \beta_3 = 2\alpha_1 + 3\alpha_2 \implies \beta_1 = \alpha_1 + 2\alpha_2$. However, $\beta_1 + \beta_2 = \alpha_1 + 3\alpha_2$ is not an inversion, so this does not give a linear π_2. Hence, we get no new elements from G_2.

For A_3, coefficient counting gives that $\beta_1, \beta_2, \beta_3$ must all be simple roots, from which $\beta_1 + \beta_2, \beta_2 + \beta_3 \in \Phi^+ \implies \beta_2 = \alpha_2$, and WLOG $\beta_1 = \alpha_1, \beta_3 = \alpha_3$. The inversions of $w \in W(A_3)$ containing a linear π_2 pattern are fully determined by this, and this determines w to be $s_2 s_1 s_3 s_2 \in W(A_3)$. One can check that this does not contain a linear π_1 and also that there is no strict subspace which contains π_2, so $s_2 s_1 s_3 s_2 \in \mathrm{red}(P_{\pi_2})/P_{\pi_1}$

Let us now show that $\mathrm{red}(P_{\pi_2})/P_{\pi_1}$ contains no elements of $W(B_3)$. Suppose we have $w \in W(B_3)$ containing a linear π_2. Let the corresponding linear map send the simple roots of A_3 to $\beta_1, \beta_2, \beta_3 \in B_3$. Let us consider the options for $\beta_1, \beta_2, \beta_3$. If the $\mathbb{R}$-span of these three roots is a proper subspace of the ambient B_3, then the restriction of w to the root system in this subspace contains π_2, so $w \notin \mathrm{red}(P_{\pi_2})$. This leaves us with the case where the $\mathbb{R}$ span of $\beta_1, \beta_2, \beta_3$ is full-dimensional. Note that $\beta_1 + \beta_2 + \beta_3 \in B_3$, so its height (the sum of its coefficients in the basis of simple roots $\alpha_1, \alpha_2, \alpha_3$ of B_3) is at most 5. It is also at least 3, and we will consider each option:

- If the height of $\beta_1 + \beta_2 + \beta_3$ is 3, the only full-dimensional option that also satisfies the condition that $\beta_1 + \beta_2, \beta_2 + \beta_3 \in \Phi^+$ is $\beta_1 = \alpha_1, \beta_2 = \alpha_2, \beta_3 = \alpha_3$ (the map in reverse order is also possible, but this gives the same inversions since π_2 is preserved under the isomorphism of A_3). Then $\beta_1 + \beta_2 + \beta_3$, $\beta_2 + \beta_3$, and (by biconvexity) $\beta_1 + 2\beta_2 + 2\beta_3$ are all inversions, so w contains a linear π_1 generated at $\beta_1 + \beta_2 + \beta_3, \beta_2 + \beta_3$, so w is not in $\mathrm{red}(P_{\pi_2})/P_{\pi_1}$.
- If the height of $\beta_1 + \beta_2 + \beta_3$ is 4, then one of $\beta_1, \beta_2, \beta_3$ has height 2, and the other two have height 1 each. B_3 only has two roots of height 2, namely $\alpha_1 + \alpha_2$ and $\alpha_2 + \alpha_3$. If the height two root is $\alpha_1 + \alpha_2$, then any root adjacent to its preimage in the A_3 Dynkin diagram has to be sent to α_3 (since neither $\alpha_1 + (\alpha_1 + \alpha_2)$ nor $(\alpha_1 + \alpha_2) + \alpha_2$ is a root). Since we are in the full-dimensional case, this implies that $\beta_2 \neq \alpha_1 + \alpha_2$, so WLOG (as before) $\beta_1 = \alpha_1 + \alpha_2$. From what we already argued, it follows that $\beta_2 = \alpha_3$. The only option for β_3 is $\beta_3 = \alpha_2$. But then w contains a linear π_1 generated at $\alpha_2 + \alpha_3, \alpha_3$. So w is not in $\mathrm{red}(P_{\pi_2})/P_{\pi_1}$. If instead the height 2 root is $\alpha_2 + \alpha_3$, then an adjacent root can be α_1 or α_3. If $\alpha_2 + \alpha_3 = \beta_2$, then WLOG $\beta_1 = \alpha_1$ and $\beta_3 = \alpha_3$. Then w contains a linear π_1 generated at $\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3$. So w is not in $\mathrm{red}(P_{\pi_2})/P_{\pi_1}$. The remaining case is that WLOG $\alpha_2 + \alpha_3 = \beta_1$, in which case either $\beta_2 = \alpha_1$, and hence $\beta_3 = \alpha_2$; or $\beta_2 = \alpha_3$, and hence $\beta_3 = \alpha_2$. The former case is impossible since then $\beta_1 + \beta_2 + \beta_3 \notin \Phi^+$. In the latter case, we find a linear π_1 generated at $\alpha_2 + \alpha_3, \alpha_3$. So w is not in $\mathrm{red}(P_{\pi_2})/P_{\pi_1}$.

- If the height of $\beta_1+\beta_2+\beta_3$ is 5, let us split into cases according to whether one of $\beta_1, \beta_2, \beta_3$ has height 3.

  If one of $\beta_1, \beta_2, \beta_3$ has height 3, then this can be either $\alpha_1+\alpha_2+\alpha_3$ or $\alpha_2+2\alpha_3$ (since B_3 has no other roots of height 3). In either case, $\beta_1+\beta_2+\beta_3 = \alpha_1+2\alpha_2+2\alpha_3$ (since this is the unique root in B_3 of height 5), and this lets us determine the other two β_i. In the former case, the other two are α_2, α_3, in which case we note by considering pairwise sums that the only option is $\beta_1 = \alpha_1+\alpha_2+\alpha_3$, $\beta_2 = \alpha_3, \beta_3 = \alpha_2$. But then w contains a linear π_1 generated at $\alpha_2+\alpha_3, \alpha_3$. So w is not in $\mathrm{red}(P_{\pi_2})/P_{\pi_1}$. In the latter case, the other two are α_1, α_2, in which case (arguing as before) we get $\beta_1 = \alpha_2+2\alpha_3, \beta_2 = \alpha_1, \beta_3 = \alpha_2$. But then there is a linear π_2 generated at $\alpha_3, \alpha_1+\alpha_2, \alpha_3$ which is contained in a 2-dimensional subspace, so $w \notin \mathrm{red}(P_{\pi_2})$.

  If there is no root among $\beta_1, \beta_2, \beta_3$ of height 3, then two have height 2 and one has height 1. The only two roots of B_3 of height 2 are $\alpha_1+\alpha_2$ and $\alpha_2+\alpha_3$, and since it is not possible that one of these appears twice (that would contradict with full-dimensionality), both must appear once among $\beta_1, \beta_2, \beta_3$. Since $(\alpha_1+\alpha_2)+(\alpha_2+\alpha_3) \notin \Phi^+$, these must be β_1 and β_3, so WLOG $\beta_1 = \alpha_1+\alpha_2, \beta_3 = \alpha_2+\alpha_3$. Using $\beta_1+\beta_2+\beta_3 = \alpha_1+2\alpha_2+2\alpha_3$, we get that $\beta_2 = \alpha_3$. But then w contains a linear π_1 generated at $\alpha_1+\alpha_2, \alpha_2+2\alpha_3$. So w is not in $\mathrm{red}(P_{\pi_2})/P_{\pi_1}$.

This completes the case check for B_3. We do the same for C_3. The case check can be set up completely analogously, and there will again be 3 options for the height of $\beta_1+\beta_2+\beta_3$. We will now find that there is exactly one $w \in W(C_3)$ that contains a linear π_2.

- If the height of $\beta_1+\beta_2+\beta_3$ is 3, then again the only option we have to consider is $\beta_1 = \alpha_1, \beta_2 = \alpha_2, \beta_3 = \alpha_3$. Then α_2, $\alpha_2+\alpha_3$, and (by biconvexity) $2\alpha_2+\alpha_3$ are all inversions, so we find a linear π_1.
- If the height of $\beta_1+\beta_2+\beta_3$ is 4, then the height 2 β_i is $\alpha_1+\alpha_2$ or $\alpha_2+\alpha_3$. If the height 2 β_i is $\alpha_1+\alpha_2$, then arguing like for B_3, we get WLOG $\beta_1 = \alpha_1+\alpha_2$, $\beta_2 = \alpha_3$, $\beta_3 = \alpha_2$. Then note that $\beta_1+\beta_2+\beta_3 = \alpha_1+2\alpha_2+\alpha_3$ is an inversion, so by biconvexity, at least one of α_1 and $2\alpha_2+\alpha_3$ is an inversion. If the former is an inversion, there is a linear π_1 generated at $\alpha_1, \alpha_2+\alpha_3$. If the latter is an inversion, there is a linear π_2 generated at $\alpha_2, \alpha_3, \alpha_2$, the span of which is a proper subspace. This leaves us with the case that the height 2 β_i is $\alpha_2+\alpha_3$. If $\alpha_2+\alpha_3 = \beta_2$, then WLOG $\beta_1 = \alpha_1$, $\beta_3 = \alpha_2$. Since $\alpha_2+\alpha_3$ is an inversion, biconvexity gives that at least one of α_2, α_3 is an inversion. If α_2 is an inversion, then we have a π_1 generated at $\alpha_2, \alpha_2+\alpha_3$. If α_3 is an inversion, then we have a π_2 generated at $\alpha_2, \alpha_3, \alpha_2$, the span of which is a proper subspace. The remaining case is that WLOG $\alpha_2+\alpha_3 = \beta_1$; then $\beta_2 = \alpha_1$ or $\beta_2 = \alpha_2$. In the case that $\beta_2 = \alpha_1$, we get that $\beta_3 = \alpha_2$, from which $\beta_1+\beta_2 = \alpha_1+\alpha_2+\alpha_3$ and $\beta_2+\beta_3 = \alpha_1+\alpha_2$ are inversions, so there is a π_1 generated at $\alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3$. In the case that $\beta_2 = \alpha_2$, we get that $\beta_3 = \alpha_1$ (the case $\beta_3 = \alpha_3$ is ruled out since $\beta_1+\beta_2+\beta_3 \in \Phi^+$). Then note that $\alpha_2, 2\alpha_2+\alpha_3, \alpha_1+\alpha_2, \alpha_1+2\alpha_2+\alpha_3$ are inversions and $\alpha_1, \alpha_2+\alpha_3$ are non-inversions (by definition of linear pattern containment). The remaining roots are $\alpha_3, \alpha_1+\alpha_2+\alpha_3, 2\alpha_1+2\alpha_2+\alpha_3$. If α_3 is an inversion, then there is a linear π_1 generated at α_2, α_3. If $\alpha_1+\alpha_2+\alpha_3$

is an inversion, then there is a linear π_1 generated at $\alpha_1+\alpha_2+\alpha_3, \alpha_2$. If $2\alpha_1+2\alpha_2+\alpha_3$ is an inversion, then there is a linear π_2 generated at $\alpha_1, 2\alpha_2+\alpha_3, \alpha_1$. So the only option is that $\alpha_3, \alpha_1+\alpha_2+\alpha_3, 2\alpha_1+2\alpha_2+\alpha_3$ are all non-inversions, in which case the inversion set is exactly known and determines w to be $s_2s_1s_3s_2 \in W(C_3)$. One can check explicitly that it does not contain any π_1, nor does it contain any π_2 in a strict subspace, so $s_2s_1s_3s_2 \in \mathrm{red}(P_{\pi_2})/P_{\pi_1}$. We have now checked all options for the height 4 case.

- If the height of $\beta_1+\beta_2+\beta_3$ is 5, we again split into cases according to whether some β_i has height 3.

  If one of $\beta_1, \beta_2, \beta_3$ has height 3, then this can be either $\alpha_1+\alpha_2+\alpha_3$ or $2\alpha_2+\alpha_3$. Either way, $\beta_1+\beta_2+\beta_3 = 2\alpha_1+2\alpha_2+\alpha_3$, from which we can deduce the other two β_i. In the case that the height 3 root is $\alpha_1+\alpha_2+\alpha_3$, we get the other two to be α_1 and α_2, from which WLOG $\beta_1=\alpha_1$, $\beta_2=\alpha_2$, $\beta_3=\alpha_1+\alpha_2+\alpha_3$. Note that $\beta_1+\beta_2+\beta_3 = 2\alpha_1+2\alpha_2+\alpha_3$ is an inversion but $\beta_3=\alpha_1+\alpha_2+\alpha_3$ is not an inversion, from which biconvexity gives that $\alpha_2+\alpha_3$ is an inversion. But then there is a π_1 generated at $\alpha_2, \alpha_2+\alpha_3$. In the case that the height 3 root is $2\alpha_2+\alpha_3$, we get that the other two are α_1 and α_1. However, these span a two-dimensional subspace.

  The remaining option is that two of $\beta_1, \beta_2, \beta_3$ have height 2. The only two height 2 roots are $\alpha_1+\alpha_2$ and $\alpha_2+\alpha_3$. By considering the dimension of the span, we see that it is not possible for one of these to appear twice, so both must be some β_i, which implies that the third root is $(2\alpha_1+2\alpha_2+\alpha_3)-(\alpha_1+\alpha_2)-(\alpha_2+\alpha_3)=\alpha_1$. By considering pairwise sums, we see that WLOG $\beta_1=\alpha_1, \beta_2=\alpha_2+\alpha_3, \beta_3=\alpha_1+\alpha_2$. Since $\alpha_2+\alpha_3$ is an inversion, biconvexity gives that at least one of α_2, α_3 is an inversion. If α_2 is an inversion, then we find a linear π_1 generated at $\alpha_2, \alpha_1+\alpha_2+\alpha_3$. If α_2 is not an inversion, then α_3 is an inversion. Also note that $\alpha_1+2\alpha_2+\alpha_3$ is an inversion but α_1 is not an inversion, so biconvexity implies that $2\alpha_2+\alpha_3$ is an inversion. So we find a linear π_2 generated at $\alpha_2, \alpha_3, \alpha_2$.

□

We proceed to present $P_3 := \mathrm{red}(P_{\pi_3})/(P_{\pi_1}\cup P_{\pi_2})$ where $\pi_3 = s_2s_1s_3s_4s_2 \in W(D_4)$.

Lemma 3.10. *The set of additional BP patterns corresponding to π_3, $P_3 = red(P_{\pi_3})/(P_{\pi_1}\cup P_{\pi_2})$, consists of*

- $s_1s_2s_1 \in W(G_2)$;
- $s_2s_1s_3s_2 \in W(B_3)$;
- $s_2s_1s_3s_4s_2 \in W(D_4)$.

One can prove this analogously to what we did for π_1 and π_2, i.e., by checking all elements of Weyl groups of irreducible root systems of rank at most 4 (which we did using computer assistance), but we skip this.

We now give the proof of Theorem 1.2 from 1.3.

Proof of Theorem 1.2. We continue with the notation $\pi_1 = s_1s_2s_1 \in W(A_2)$, $\pi_2 = s_2s_1s_3s_2 \in W(A_3)$, and $\pi_3 = s_2s_1s_3s_4s_2 \in W(D_4)$. By Theorem 1.3, for an irreducible root system Φ, $w \in W(\Phi)$ is boolean iff it avoids the linear patterns π_1, π_2, π_3. By lemma 3.7 and the observations after it, w avoids linear π_1, π_2, π_3 iff

w avoids all BP patterns in $P_1 \cup P_2 \cup P_3 =: P$, which is exactly the set of BP patterns given in Theorem 1.2. Hence, for Φ irreducible, $w \in W(\Phi)$ is boolean iff it avoids all BP patterns in P. Note that for any (not necessarily irreducible) root system Φ, $w \in W(\Phi)$ is boolean iff the restrictions of w to all irreducible components of Φ are boolean. Note that if w avoids all the BP patterns in P, then the restriction of w to any irreducible component also avoids all BP patterns in P, and hence is boolean. Conversely, if w contains a BP pattern in P, then note that since all these patterns live in irreducible root systems, the subspace in which such a pattern is found is a subspace of an irreducible component, and hence the restriction of w to this irreducible component is not boolean. These two directions together show that $w \in W(\Phi)$ is boolean iff it avoids all BP patterns in P. This is the statement of 1.2, which is what we wanted to prove. □

4. k-Boolean permutations

Inspired by Proposition 3.1, we define k-boolean permutations.

Definition 4.1. A permutation $w \in \mathfrak{S}_n$ is *k-boolean* if for any reduced word of w, there is no simple transposition s_i that appears strictly more than k times.

We see that w is 0-boolean if and only if w is the identity. Also by definition, being 1-boolean is the same as being boolean.

Theorem 4.2. *A permutation $w \in \mathfrak{S}_n$ is 2-boolean if and only if w avoids 3421, 4312, 4321 and 456123.*

Remark 4.3. It is clear that being 0-boolean is equivalent to avoiding the pattern 21, and we know that being 1-boolean and being 2-boolean are characterized by pattern avoidance as well. However, it is not true that being k-boolean for $k \geq 3$ is characterized by pattern avoidance. We have that $436512 = s_3s_2s_3s_4s_5s_1s_2s_3s_4s_3$ is not 3-boolean since this reduced expression contains 4 copies of s_3. However, 4357612, which contains 436512 as a pattern, is 3-boolean by a computer check.

We prove Theorem 4.2 in Section 4.1 and we then enumerate them in Section 4.2.

4.1. **Proof of Theorem 4.2.**

This section is devoted to proving Theorem 4.2, which boils down to tedious case checking. We split the proof into two halves, one for each direction.

Lemma 4.4. *If a permutation $w \in \mathfrak{S}_n$ contains 3421, 4312, 4321 or 456123, then there exists a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ where some simple transposition $s_k = (k\ k+1)$ appears at least 3 times.*

Proof. Multiplying w by s_k on the right can be thought of as swapping the values at index k and $k+1$, in one-line notation of the permutation. We are going to construct a reduced expression of w by using the simple transposition to gradually reduce the length of w until we obtain the identity permutation.

Case 1: w contains 3421. Suppose that w contains 3421 at indices $r_1 < r_2 < r_3 < r_4$ with $w(r_1) = c$, $w(r_2) = d$, $w(r_3) = b$, $w(r_4) = a$ with $a < b < c < d$. We pick (r_1, r_2, r_3, r_4) such that $r_3 - r_2$ is as small as possible. In this way, for every j in the range of $r_2 < j < r_3$, if $w(j) > c$, then we can replace r_2 by j to decrease $r_3 - r_2$, contradicting its minimality and if $a < w(j) < c$, we can replace r_3 by j to decrease

$r_3 - r_2$, contradicting its minimality as well. As a result, for $r_2 < j < r_3$, we must have $w(j) < a$. Let $k = r_3 - 1$ and we will show that we can use s_k at least 3 times to decrease w down to the identity.

First, let $w^{(1)} = ws_{r_2}s_{r_2+1}\cdots s_{r_3-2}$ where the length of w is decreasing by 1 at each step. We then have $w^{(1)}(k) = d$ and $w^{(1)}(k+1) = b$, which form a descent.

Let $w^{(2)} = w^{(1)}s_k$.

Now we multiply $w^{(2)}$ by some products of s_i's to obtain $w^{(3)}$, where $k+1 \leq i \leq r_4 - 1$ to sort the indices $\{k+1, k+2, \ldots, r_4\}$, i.e., $w^{(3)}(k+1) < \cdots < w^{(3)}(r_4)$ and $\{w^{(3)}(k+1), \ldots, w^{(3)}(r_4)\} = \{w^{(2)}(k+1), \ldots, w^{(2)}(r_4)\}$, while decreasing the length of w by 1 in each step.

We observe that $w^{(3)}(k) = b$ and $w^{(3)}(k+1) = a' \leq w^{(2)}(r_4) = a$ so let $w^{(4)} = w^{(3)}s_k$. Finally, notice that $w^{(4)}(r_1) = c > w^{(4)}(k+1) = b$, which means $w^{(4)}$ has an inversion supported on s_k. By Lemma 3.2 (Remark 3.3), any reduced expression of $w^{(4)}$ contains s_k. We have thus obtained three copies of s_k.

A diagram of the above steps is shown in Figure 2.

$$\begin{array}{rll}
w &= \cdots c \cdots d \cdots b \cdots\cdots a \cdots & \\
w^{(1)} &= \cdots c \cdots\cdots db \cdots\cdots a \cdots & \big\downarrow s_k \\
w^{(2)} &= \cdots c \cdots\cdots bd \cdots\cdots a \cdots & \\
w^{(3)} &= \cdots c \cdots\cdots ba' \cdots d \cdots\cdots & \big\downarrow s_k \\
w^{(4)} &= \cdots \underbrace{c \cdots\cdots a'b}_{\text{supported on } s_k} \cdots d \cdots\cdots &
\end{array}$$

FIGURE 2. 3421 implies some s_k appearing at least 3 times

Case 2: w contains 4312. Since 4312 is the inverse of 3421, this case follows from Case 1 by taking inverse.

Case 3: w contains 4321. This is a simpler version of Case 1. As we have already done Case 1, we may as well assume that w avoids 3421. Suppose that w contains 4321 at indices $r_1 < r_2 < r_3 < r_4$ with $w(r_1) = d$, $w(r_2) = c$, $w(r_3) = b$, $w(r_4) = a$ with $a < b < c < d$. For j in the range of $r_2 < j < r_3$, we must have $w(j) < w(r_2) = c$ since otherwise, w contains 3421 at indices $r_2 < j < r_3 < r_4$. We can now run exactly the same argument as in Case 1 by switching all c's with d's. One could also refer to Figure 2 by considering c and d swapped.

Case 4: w contains 456123. The argument is also largely similar. Suppose that w contains 456123 at indices $r_1 < \cdots < r_6$ with $w(r_1) = d$, $w(r_2) = e$, $w(r_3) = f$, $w(r_4) = a$, $w(r_5) = b$ and $w(r_6) = c$. Let $r_3 \leq k < r_4$ be any index in between r_3 and r_4. Consider $w^{(1)}$, which is obtained from w by sorting indices $r_3, r_3+1, \ldots, r_4$ in order, i.e. $w^{(1)}(r_3) < \cdots < w^{(1)}(r_4)$. Equivalently, we can obtain $w^{(1)}$ from w by multiplying s_j on the right, for some $r_3 \leq j < r_4$, so that the length decreases after the multiplication, until such operation cannot be performed anymore. By Lemma 3.2 (Remark 3.3), as $w(r_3) > w(r_4)$, s_k must be used. Next, let $w^{(2)}$ be the permutation obtained from $w^{(1)}$ by sorting indices $r_2, \ldots, r_5$. Similarly, as $w^{(1)}(r_2) > w^{(1)}(r_5)$, s_k is used in the process. Finally, let $w^{(3)}$ be the permutation obtained from $w^{(2)}$ by sorting indices $r_1, \dots, r_6$ and as $w^{(2)}(r_1) > w^{(2)}(r_6)$, s_k is used a third time. $\square$

We now proceed to the other direction of Theorem 4.2.

Lemma 4.5. *Let w be a permutation that contains one of 3421, 4312, 4321, 456123. If $u = ws_k$, or $u = s_k w$, such that $\ell(u) = \ell(w) + 1$, then u also contains one of these patterns.*

Proof. Let's note that the set of patterns of interest is closed under taking inverses, so it suffices to consider only the case $u = ws_k$. Assume that w contains π, one of the pattern of interst, at indices $r_1 < \cdots < r_m$, where $m \in \{4, 6\}$. If $\{k, k+1\} \cap \{r_1, \dots, r_m\} \leq 1$, then $u = ws_k$ contains the same pattern π. If $\{k, k+1\} \subset \{r_1, \dots, r_m\}$, then u contains πs_j, for some s_j such that $\ell(\pi s_j) = \ell(\pi) + 1$. If $\pi = 3421$, then we must have $j = 1$ and $\pi s_j = 4321$; if $\pi = 4312$, then $j = 3$ and $\pi s_j = 4321$; if $\pi = 4321$, no such j exists and we have a contradiction. The remaining case is $\pi = 456123$ and if $j = 1$, then u contains 546123, which contains 4312; if $j = 2$, then u contains 465123, which contains 4312; if $j = 4$, then u contains 456213, which contains 3421; if $j = 5$, then u contains 456132, which contains 3421. $\square$

Lemma 4.6. *Let $w \in \mathfrak{S}_n$ be a permutation with a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ where some simple transposition s_k appears at least 3 times, then w contains one of 3421, 4312, 4321, 456123.*

Proof. Use induction on $\ell(w)$. If $s_{i_1} \neq s_k$, then $w' = s_{i_2} \cdots s_{i_\ell}$ contains s_k at least 3 times so by induction hypothesis, w' contains one of the 3421, 4312, 4321, 456123. By Lemma 4.5, w contains one of the patterns as well and we are done. Thus, we can assume that $s_{i_1} = s_k$, and similarly $s_\ell = s_k$, so that $w = s_k \cdots s_k \cdots s_k$.

As w has a right inversion s_k, $w(k) > w(k+1)$. Let $x = w(k+1)$ and $y = w(k)$ with $x < y$. As w has a left inversion s_k, we know that $k+1$ appears before k in w. Let $w(i) = k+1$ and $w(j) = k$ with $i < j$. If $\{i, j\} = \{k, k+1\}$, then $w(k) = k+1$ and $w(k+1) = w(k)$, so that $ws_k = s_k w$ and w cannot possibly have a reduced expression starting and ending at s_k. This case is impossible. We will consider various orderings of $i, j, k, k+1$ and $x, y, k, k+1$ to find patterns in w. Write $u = s_k w s_k$ so that $\ell(u) = \ell(w) - 2$. By Lemma 3.2, since a reduced expression of u contains s_k, u has an inversion across index k. We are going to use this strategy for the following cases.

Case 1: $|\{i, j\} \cup \{k, k+1\}| = 3$. We have a few subcases here.

If $i = k$, then $j > k+1$. As there are at least two values among $\{w(k+1), w(k+2), \dots, w(n)\}$ that are at most k, namely $w(k+1) < w(k) = k+1$ and $w(j) = k$, there must be at least two values $\{w(1), \dots, w(k)\}$ that are greater than k. We already have $w(k) = k+1$ so there exists some $a < k$ such that $w(a) > k$. But $w(a) \neq k+1$ so $w(a) \geq k+2$. As a result, w contains 4312 at indices $a, k, k+1, j$.

If $i = k+1$, then $j > k+1$ and we see that $w(k) > w(k+1) > w(j)$. Then $u(k) = k$, $u(k+1) = w(k) > w(k+1) = k+1$, $u(j) = k+1$. Since u has an inversion across index k, we must have some $a \in \{k+1, \dots, n\}$ such that $u(a) \leq k$. As $u(k) = k$, $u(a) < k$, $a \neq k, k+1, j$. We see that $w(a) = u(a)$, and if $a < j$, w contains 4312 at indices $k < k+1 < a < j$ and if $a > j$, w contains 4321 at indices $k < k+1 < j < a$.

If $j = k$, then $i < k$ and $w(i) > w(k) > w(k+1)$. Similar as above, we see that $u(i) = k$, $u(k) = w(k+1) < w(k) = k$, $u(k+1) = k+1$. As u has s_k in its reduced expressions, there exists some $a \in \{1,\ldots,k\}$ such that $u(a) > k$. Thus, $a \neq i, k$ and $u(a) \geq k+2$. Back to w, we have $w(a) = u(a)$. So if $a < i$, w contains 4321 at indices $a < i < k < k+1$ and if $a > i$, w contains 3421 at indices $i < a < k < k+1$.

If $j = k+1$, then $i < k$. Both $w(i), w(k)$ are greater than k. As $\{w(1),\ldots,w(k)\} \cap \{k+1,\ldots,n\}$ has cardinality at least 2, $\{w(k+1),\ldots,w(n)\} \cap \{1,\ldots,k\}$ has cardinality at least 2. So there exists some $a > k+1$ such that $w(a) < k$. As a result, w contains 3421 at indices $i < k < k+1 < a$.

The situation when $|\{x,y\} \cup \{k,k+1\}| = 3$ can be deduced from Case 1 by taking inverses. From now on, assume that both $\{i,j\}$ and $\{x,y\}$ are disjoint from $\{k,k+1\}$. Table 3 shows how we divide the problem into cases.

	$x<y<k<k{+}1$	$x<k<k{+}1<y$	$k<k{+}1<x<y$
$i<j<k<k{+}1$	Case 2 (4321)	Case 3 (4321/4312)	Case 3 (4321/4312)
$i<k<k{+}1<j$	Case 3 (4321/3421)	Case 5 (...)	Case 4 (4321/4312)
$k<k{+}1<i<j$	Case 3 (4321/3421)	Case 4 (4321/3421)	Case 2 (4321)

TABLE 3. Cases for the proof of Lemma 4.6

Case 2: $i<j<k<k{+}1$ and $x<y<k<k{+}1$ or $k<k{+}1<i<j$ and $k<k{+}1<x<y$. In this case, we directly see that w contains 4321 at indices $i,j,k,k{+}1$ (either $i<j<k<k{+}1$ or $k<k{+}1<i<j$).

Case 3: $i<j<k<k{+}1$ and $y > k+1$. Since $\{w(1),\ldots,w(k)\} \cap \{k+1,\ldots,n\}$ has cardinality at least 2, namely $w(i) = k+1$ and $w(k) = y > k+1$, $\{w(k+1),\ldots,w(n)\} \cap \{1,\ldots,k\}$ must have cardinality at least 2. Say $k < a < b$ and $w(a), w(b) \leq k$. As $w(j) = k$ with $j < k$, we must have $w(a), w(b) < k$. As a result, w contains either 4321 or 4312 at indices $i < j < a < b$. By taking inverses, we are also down with the case where $x<y<k<k{+}1$ and $j > k+1$.

Case 4: $i<k<k{+}1<j$ and $k<k{+}1<x< y$. Since $\{w(1),\ldots,w(k)\} \cap \{k+1,\ldots,n\}$ has cardinality at least 2, namely $w(i) = k+1$ and $w(k) = y > k+1$, $\{w(k+1),\ldots,w(n)\} \cap \{1,\ldots,k\}$ must have cardinality at least 2. Besides $w(j) = k$, we must some $a > k$, $a \neq j$, such that $w(a) < k$. Also $a > k+1$ since $w(k+1) = x > k+1$. As a result, w contains 4321 at indices $k, k+1, j, a$ if $a > j$ and contains 4312 at indices $k, k+1, a, j$ if $a < j$.

Case 5: $i<k<k{+}1<j$ and $x<k<k{+}1<y$. Recall that $u = s_k w s_k$. In this case, $u(i) = k$, $u(k) = w(k+1) = x < k$, $u(k+1) = w(k) = y > k+1$, $u(j) = k+1$. Since a reduced expression of u uses s_k, we cannot possibly have $\{u(1),\ldots,u(k)\} = \{1,\ldots,k\}$. There exists $a < k$ such that $u(a) > k$ and $b > k$ such that $u(b) < k$. Since $u(j) = k+1$, $u(a) > k+1$, and also $a \neq i, k$. Similarly, $u(b) < k$ and $b \neq k+1, j$. This also tells us $u(a) = w(a)$, $u(b) = w(b)$. If $a < i$, then w contains 4312 at indices $a, i, k+1, j$ and if $w(a) > y$, then w contains 4312 at indices $a, k, k+1, j$. Similarly, if $b > j$, then w contains 3421 at indices i, k, j, b and if $w(b) < x$, then w contains 3421 at indices $i, k, k+1, b$. The final remaining case is that $i < a < k$, $k+1 < w(a) < y$, $k+1 < b < j$, $x < w(b) < k$, where w contains 456123 at indices $i, a, k, k+1, b, j$. $\square$

Now Theorem 4.2 follows from Lemma 4.4 and Lemma 4.6.

4.2. **Enumeration of 2-boolean permutations.** Throughout this section, let $f(n)$ denote the number of 2-boolean permutations in $\mathfrak{S}_n$. We adopt the convention that $f(0) = 1$. We have that $f(1) = 1$, $f(2) = 2$, $f(3) = 6$, $f(4) = 21$, $f(5) = 78$, and so on, which appears as sequence A124292 in OEIS [8].

Theorem 4.7. *Let $f(n)$ be the number of 2-boolean permutations in $\mathfrak{S}_n$. Then*

$$\sum_{n\geq 0} f(n)q^n = \frac{1-5q+5q^2}{1-6q+9q^2-3q^3}.$$

In this section, we think of 2-boolean permutations as permutations that avoid 3421, 4312, 4321 and 456123 (Theorem 4.2). Let's first look at what a typical 2-boolean permutation looks like. Let w be 2-boolean. If $w(1) = 1$, then w restricted to indices $2, 3, \ldots, n$ is just a 2-boolean permutation in $\mathfrak{S}_{n-1}$ (and it is easy to see that this can in fact be any 2-boolean permutation in $\mathfrak{S}_{n-1}$). If $w(1) \neq 1$, we define the following sets:

$$\begin{aligned}
C(w) =&\{(i, w(i)) \mid 1 < i < w^{-1}(1), 1 < w(i) < w(1)\},\\
A(w) =&\{(i, w(i) \mid 1 < i < w^{-1}(1), w(i) > w(1))\},\\
B(w) =&\{(i, w(i)) \mid i > w^{-1}(1), 1 < w(i) < w(1)\}.
\end{aligned}$$

Write $a(w) = |A(w)|$, $b(w) = |B(w)|$ and $c(w) = |C(w)|$ for cardinality. Note that all these quantities are only defined for those w such that $w(1) \neq 1$. See Figure 3 for a visual description of these regions. Since w avoids 4321, we see

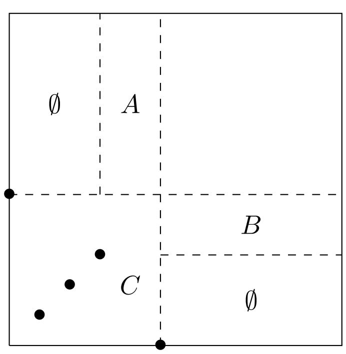


FIGURE 3. Structure of a 2-boolean permutation

that entries in $C(w)$ must be increasing. Let $C(w) = \{(i_1, w(i_1)), \ldots, (i_c, w(i_c))\}$ with $i_1 < \cdots < i_c$ and $w(i_1) < \cdots < w(i_1)$. As w avoids 3421, the region $\{(i, w(i)) \mid 1 \leq i \leq i_c, w(i) > w(1)\}$ must be empty. Similarly as w avoids 4312, the region $\{(i, w(i)) | i > w^{-1}(1), 1 < w(i) < w(i_c)\}$ is empty. These empty sets are indicated in Figure 3. Consequently, we know that $C = \{(2, 2), (3, 3), \ldots, (c+1, c+1)\}$ where $c = c(w)$. Moreover, it is impossible for $a(w) \geq 2$ and $b(w) \geq 2$ to happen simultaneously. Otherwise, say $A(w)$ contains $(x_1, w(x_1))$ and $(x_2, w(x_2))$ while $B(w)$ contains $(y_1, w(y_1))$ and $(y_2, w(y_2))$ with $x_1 < x_2$ and $y_1 < y_2$. If $w(x_1) > w(x_2)$, then w contains 4312 at indices x_1, x_2, w^{-1}, y_1 and similarly if $w(y_1) > w(y_2)$, w contains 3421 at indices $1, x_1, y_1, y_2$; and if finally $w(x_1) < w(x_2)$

and $w(y_1) < w(y_2)$, then w contains 456123 at indices $1, x_1, x_2, w^{-1}(1), y_1, y_2$. As a result, either $a(w) \leq 1$ or $b(w) \leq 1$ for a 2-boolean permutation w.

As an important piece of notation, we use $f_c^{a,b}(n)$ to denote the number of 2-boolean permutations w in $\mathfrak{S}_n$ such that $a(w) = a$, $b(w) = b$ and $c(w) = c$. Note that $f_c^{a,b}(n) = f_c^{b,a}(n)$ by the symmetry of taking inverses. We will also omit some superscripts or subscripts to mean we require less conditions. For example, $f^a(n)$ is the number of 2-boolean permutations w in $\mathfrak{S}_n$ with $a(w) = a$.

The following lemma is the key to our recurrence.

Lemma 4.8. *We have the following identities for $n \geq 4$:*

$$f^0(n) = \sum_{1 \leq k \leq n-1} f(k), \tag{1}$$

$$f^1(n) = f(n-1) - f(n-2), \tag{2}$$

$$f^{0,0}(n) = \sum_{0 \leq k \leq n-2} f(k), \tag{3}$$

$$f^{0,1}(n) = \sum_{1 \leq k \leq n-2} f(k), \tag{4}$$

$$f^{1,1}(n) = f(n-2) - 1. \tag{5}$$

Proof. We will start by proving (1). Note that partitioning 2-boolean permutations according to the value of c, we obtain

$$f^0(n) = \sum_{c=0}^{n-2} f_c^0(n).$$

We will now show that $f_c^0(n) = f(n-1-c)$. Consider a 2-boolean permutation $w \in \mathfrak{S}_n$ with $c(w) = c$ and $a(w) = 0$. Let w' be the restriction of w to the indices $1, c+3, c+4, \ldots, n$. Note that $w' \in \mathfrak{S}_{n-1-c}$ is a 2-boolean permutation. Furthermore, this map $w \to w'$ takes different 2-boolean w with $c(w) = c$ to different 2-boolean $w' \in \mathfrak{S}_{n-1-c}$. Also note that for any 2-boolean $w' \in \mathfrak{S}_{n-1-c}$, if we construct a permutation $w \in \mathfrak{S}_n$ by letting $w(2) = 2, w(3) = 3, \ldots, w(c+1) = c+1$, $w(c+2) = 1$, and we let the restriction of w to the indices $1, c+3, c+4, \ldots, n$ be equal to w', then w is also 2-boolean with $a(w) = 0$ and $c(w) = c$. To see this, observe that for w to contain 3421, 4312, 4321, or 456123, some image $\leq c+1$ would have to be 3, 4, or 5 in the pattern, which is impossible. This map is also injective. Hence, we have a bijection showing that $f_c^0(n) = f(n-1-c)$. This lets us rewrite the above sum as

$$f^0(n) = \sum_{c=0}^{n-2} f_c^0(n) = \sum_{c=0}^{n-2} f(n-1-c) = \sum_{k=1}^{n-1} f(k),$$

The proof of (3) is completely analogous to the proof of (1), with the only difference being that we instead consider the restriction of w to the indices $c+3, c+4, \ldots, n$, and note that this can be any 2-boolean $w' \in \mathfrak{S}_{n-2-c}$, whereas $w(1) = c+2$, $w(2) = 2, w(3) = 3, \ldots, w(c+1) = c+1, w(c+2) = 1$. The fourth identity is also analogous, with the restriction being to the same indices $c+3, c+4, \ldots, n$, and the fixed values being $w(1) = c+3$, $w(2) = 2, \ldots, w(c+1) = c+1$.

As for (2), consider a 2-boolean $w \in \mathfrak{S}_n$ with $a(w) = 1$ and $c(w) = c$. Then $w(2) = 2, w(3) = 3, \ldots, w(c+1) = c+1$, and $w(c+3) = 1$. The restriction of

w to the rest of the indices $1, c+2, c+4, c+5, \ldots, n$ is a 2-boolean permutation $w' \in \mathfrak{S}_{n-c-1}$ with $w'(1) < w'(2)$. Furthermore, any such permutation w' can be inserted to these indices while giving a 2-boolean w. These maps are inverses of each other, so it suffices to count the number of such permutations. For this, it suffices to count the size of the complement, i.e. the number of 2-boolean permutations $u \in \mathfrak{S}_{n-c-1}$ with $u(1) > u(2)$. This is equivalent to $u(1) > 1$ and $c(u) \geq 1$ or $c(u) = 0, a(u) = 0$.

For the first case, i.e. that $u(1) > 1$ and $c(u) \geq 1$, we can count the number of such $u \in \mathfrak{S}_{n-c-1}$ in the following way. Note that $u(2) = 2$ (since $c(u) \geq 1$, and the restriction of u to the rest of the indices $1, 3, 4, \ldots, n-c-1$ is a 2-boolean permutation $u' \in \mathfrak{S}_{n-c-2}$ such that $u(1) > 1$. Furthermore, when we insert any such permutation to these indices, no bad pattern is created that involves the index 2. These maps are clearly inverses of each other, so we have a bijection. The number of 2-boolean $u' \in \mathfrak{S}_{n-c-2}$ with $u(1) > 1$ is $f(n-c-2) - f(n-c-3)$. By our bijection, this is also the number of 2-boolean $u \in \mathfrak{S}_{n-c-1}$ with $u(1) > 1$ and $c(u) \geq 1$.

For the second case, i.e. that $u(1) > 1, c(u) = 0$, and $a(u) = 0$, the number of such $u \in \mathfrak{S}_{n-c-1}$ is $f_0^0(n-c-1)$, which is equal to $f(n-c-2)$ as argued before.

Putting everything together and summing over c, we get that the number of 2-boolean w with $a(w) = 1$ is

$$\begin{aligned} f^1(n) &= \sum_{c=0}^{n-3} f(n-c-1) - ((f(n-c-2) - f(n-c-3)) + f(n-c-2)) \\ &= \sum_{c=0}^{n-3} f(n-c-1) - 2f(n-c-2) + f(n-c-3). \end{aligned}$$

This sum telescopes, and we are left with the desired

$$f^1(n) = f(n-1) - f(n-2) - f(0) + f(1) = f(n-1) - f(n-2).$$

It remains to show (5). Consider a permutation w with $a(w) = b(w) = 1$, and $c(w) = c$. Then $w(1) = c+3$, $w(2) = 2, w(3) = 3, \ldots w(c+1) = c+1$, and $w(c+3) = 1$. Let w' be the restriction of w to the rest of the indices $c+2, c+4, c+5, \ldots, n$. Then w' is a 2-boolean permutation in S_{n-c-2} with $w'(1) \neq 1$. Conversely, when we insert any such permutation to these indices, no bad pattern is created, and the result is w with $a(w) = b(w) = 1, c(w) = c$. This gives a bijection showing that $f_c^{1,1}(n) = f(n-c-2) - f(n-c-3)$. We sum over c:

$$f^{1,1}(n) = \sum_{c=0}^{c=n-4} f_c^{1,1}(n) = \sum_{c=0}^{c=n-4} f(n-c-2) - f(n-c-3).$$

This sum telescopes, and we are left with the desired identity

$$f^{1,1}(n) = f(n-2) - f(1) = f(n-2) - 1.$$

□

We are now ready to finish the proof of Theorem 4.7.

Proof of Theorem 4.7. Let $n \geq 4$. Recall that for a 2-boolean permutation $w \in \mathfrak{S}_n$ with $w(1) \neq 1$, we have either $a(w) \leq 1$ or $b(w) \leq 1$. By the symmetry between

$A(w)$ and $B(w)$, and by simple inclusion-exclusion, we obtain

$$f(n) = f(n-1) + 2f^0(n) + 2f^1(n) - f^{0,0}(n) - 2f^{0,1}(n) - f^{1,1}(n)$$

where the term $f(n-1)$ accounts for those permutations w with $w(1) = 1$ while $f^0(n)$ accounts for those with $a(w) = 0$ and another $f^0(n)$ corresponds to those with $b(w) = 0$ and so on. For simplicity of notation, write $S = \sum_{k=1}^{n-3} f(k)$. By Lemma 4.8, we continue the computation

$$\begin{aligned} f(n) =& f(n-1) + 2(f(n-1) + f(n-2) + S) + 2(f(n-1) - f(n-2)) \\ & - (f(n-2) + S + 1) - 2(f(n-2) + S) - (f(n-2) - 1) \\ =& 5f(n-1) - 4f(n-2) - S + 1. \end{aligned}$$

For $n \geq 5$, we write the above equation using $n-1$ to get

$$f(n-1) = 5f(n-2) - 4f(n-3) - \sum_{k=1}^{n-4} f(k) + 1.$$

Subtract from the above computation, we obtain a linear recurrence

$$f(n) - 6f(n-1) + 9f(n-2) - 3f(n-3) = 0$$

for $n \geq 5$. Together with the initial terms $f(0) = f(1) = 1$, $f(2) = 2$, $f(3) = 6$ and $f(4) = 21$, we obtain the desired generating function. $\square$

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